CHAPTER 12

Analysis of Indeterminate Beams and Frames by the Slope-Deflection Method

12.1 Introduction

The *slope-deflection method* is a procedure for analyzing indeterminate beams and frames. It is known as a *displacement method* since *equilibrium equations*, which are used in the analysis, are expressed in terms of *unknown joint displacements*.

The slope-deflection method is important because it introduces the student to the *stiffness method* of analysis. This method is the basis of many general-purpose computer programs for analyzing all types of structures—beams, trusses, shells, and so forth. In addition, *moment distribution*—a commonly used hand method for analyzing beams and frames rapidly—is also based on the stiffness formulation.

In the slope-deflection method an expression, called the *slope-deflection equation*, is used to relate the moment at each end of a member both to the end displacements of the member and to the loads applied to the member between its ends. End displacements of a member can include both a rotation and a translation perpendicular to the member's longitudinal axis.

12.2 Illustration of the Slope-Deflection Method

To introduce the main features of the slope-deflection method, we briefly outline the analysis of a two-span continuous beam. As shown in Figure 12.1*a*, the structure consists of a single member supported by rollers at points *A* and *B* and a pin at *C*. We imagine that the structure can be divided into beam segments *AB* and *BC* and joints *A*, *B*, and *C* by passing planes through the beam an infinitesimal distance before and after each support (see Fig. 12.1*b*). Since the joints are essentially points in space, the

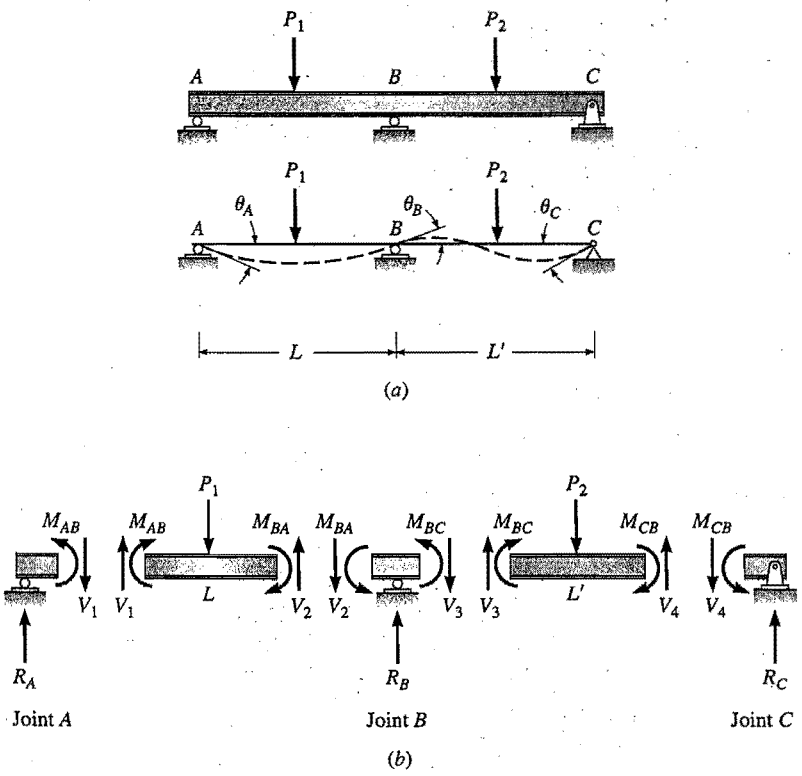

Figure 12.1: (*a*) Continuous beam with applied loads (deflected shape shown by dashed line); (*b*) free bodies of joints and beams (sign convention: clockwise moment on the end of a member is positive).

length of each member is equal to the distance between joints. In this problem θ_A, θ_B, and θ_C, the rotational displacements of the joints (and also the rotational displacements of the ends of the members), are the unknowns. These displacements are shown to an exaggerated scale by the dashed line in Figure 12.1*a*. Since the supports do not move vertically, the lateral displacements of the joints are zero; thus there are no unknown joint translations in this example.

To begin the analysis of the beam by the slope-deflection method, we use the *slope-deflection equation* (which we will derive shortly) to express the moments at the ends of each member in terms of the unknown joint displacements and the applied loads. We can represent this step by the following set of equations:

$$\begin{aligned} M_{AB} &= f(\theta_A,\ \theta_B,\ P_1) \\ M_{BA} &= f(\theta_A,\ \theta_B,\ P_1) \\ M_{BC} &= f(\theta_B,\ \theta_C,\ P_2) \\ M_{CB} &= f(\theta_B,\ \theta_C,\ P_2) \end{aligned} \tag{12.1}$$

where the symbol $f(\ )$ stands for *a function of.*

We next write equilibrium equations that express the condition that the joints are in equilibrium with respect to the applied moments; that is, the sum of the moments applied to each joint by the ends of the beams framing into the joint equals zero. As a sign convention we assume that all *unknown moments* are *positive* and act *clockwise* on the *ends of members*. Since the moments applied to the ends of members represent the action of the joint on the member, equal and oppositely directed moments must act on the joints (see Fig. 12.1*b*). The three joint equilibrium equations are

$$\begin{aligned} &\text{At joint } A: && M_{AB} = 0 \\ &\text{At joint } B: && M_{BA} + M_{BC} = 0 \\ &\text{At joint } C: && M_{CB} = 0 \end{aligned} \tag{12.2}$$

By substituting Equations 12.1 into Equations 12.2, we produce three equations that are functions of the three unknown displacements (as well as the applied loads and properties of the members that are specified). These three equations can then be solved simultaneously for the values of the unknown joint rotations. After the joint rotations are computed, we can evaluate the member end moments by substituting the values of the joint rotations into Equations 12.1. Once the magnitude and direction of the end moments are established, we apply the equations of statics to free bodies of the beams to compute the end shears. As a final step, we compute the support reactions by considering the equilibrium of the joints (i.e., summing forces in the vertical direction).

In Section 12.3 we derive the slope-deflection equation for a typical flexural member of constant cross section using the moment-area method developed in Chapter 9.

12.3 Derivation of the Slope-Deflection Equation

To develop the slope-deflection equation, which relates the moments at the ends of members to the end displacements and the applied loads, we will analyze span AB of the continuous beam in Figure 12.2*a*. Since differential settlements of supports in continuous members also create end moments, we will include this effect in the derivation. The beam, which is initially straight, has a constant cross section; that is, EI is constant along the longitudinal axis. When the distributed load $w(x)$, which can vary in any arbitrary manner along the beam's axis, is applied, supports A and B settle, respectively, by amounts Δ_A and Δ_B to points A' and B'. Figure 12.2*b* shows a free body of span AB with all applied loads. The moments M_{AB} and M_{BA} and the shears V_A and V_B represent the forces exerted by the joints on the ends of the beam. Although we assume that no axial load acts, the presence of small to moderate values of axial load (say, 10 to 15

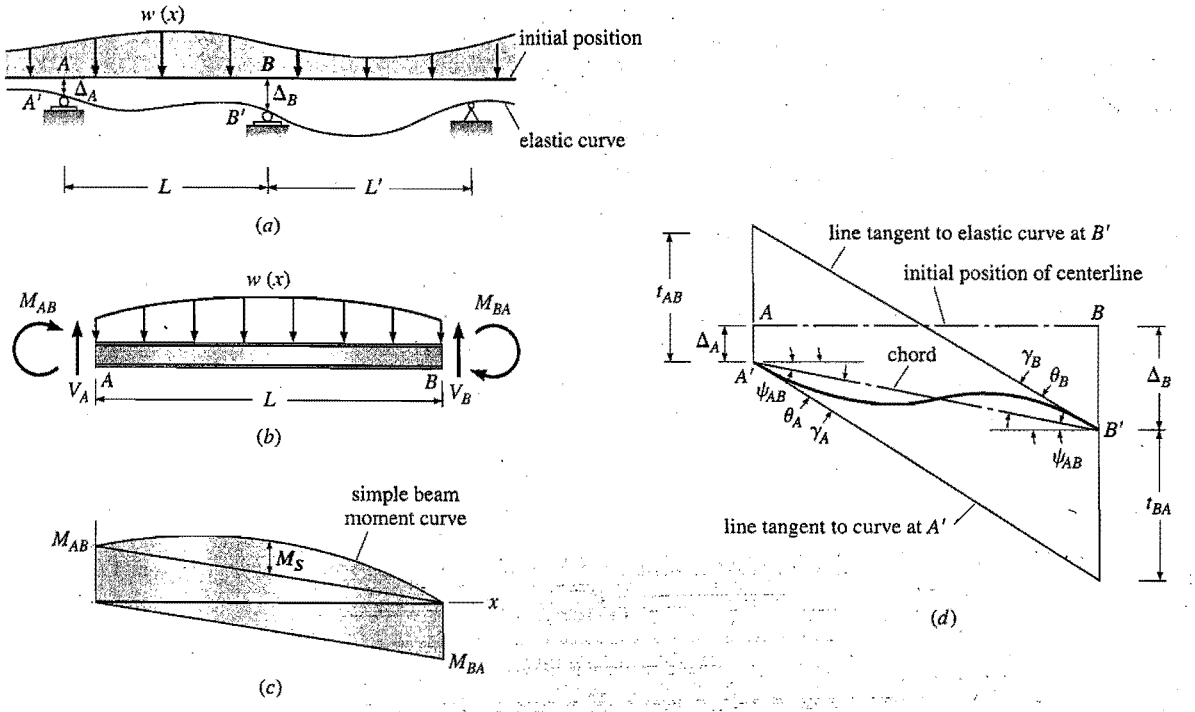

Figure 12.2: (*a*) Continuous beam whose supports settle under load; (*b*) free body of member *AB*; (*c*) moment curve plotted by parts, M_S equals the ordinate of the simple beam moment curve; (*d*) deformations of member *AB* plotted to an exaggerated vertical scale.

percent of the member's buckling load) would not invalidate the derivation. On the other hand, a large compression force would reduce the member's flexural stiffness by creating additional deflection due to the secondary moments produced by the eccentricity of the axial load—the *P*-Δ effect. As a sign convention, we assume that moments acting at the ends of members in the *clockwise direction* are *positive*. Clockwise rotations of the ends of members will also be considered positive.

In Figure 12.2*c* the moment curves produced by both the distributed load $w(x)$ and the end moments M_{AB} and M_{BA} are drawn by parts. The moment curve associated with the distributed load is called the *simple beam moment curve*. In other words, in Figure 12.2*c*, we are superimposing the moments produced by three loads: (1) the end moment M_{AB}, (2) the end moment M_{BA}, and (3) the load $w(x)$ applied between ends of the beam. The moment curve for each force has been plotted on the side of the beam that is placed in compression by that particular force.

Figure 12.2*d* shows the deflected shape of span *AB* to an *exaggerated scale*. All angles and rotations are shown in the positive sense; that is, all have undergone clockwise rotations from the original horizontal position of the axis. The slope of the chord, which connects the ends of the member at points *A'* and *B'* in their deflected position, is denoted by ψ_{AB}. To establish if a chord angle is positive or negative, we can draw a horizontal line through either end of the beam. If the horizontal line must be

rotated clockwise through an acute angle to make it coincide with the chord, the slope angle is positive. If a counterclockwise rotation is required, the slope is negative. Notice, in Figure 12.2*d*, that ψ_{AB} is positive regardless of the end of the beam at which it is evaluated. And θ_A and θ_B represent the end rotations of the member. At each end of span *AB*, tangent lines are drawn to the elastic curve; t_{AB} and t_{BA} are the tangential deviations (the vertical distance) from the tangent lines to the elastic curve.

To derive the slope-deflection equation, we will now use the second moment-area theorem to establish the relationship between the member end moments M_{AB} and M_{BA} and the rotational deformations of the elastic curve shown to an exaggerated scale in Figure 12.2*d*. Since the deformations are small, γ_A, the angle between the chord and the line tangent to the elastic curve at point *A*, can be expressed as

$$\gamma_A = \frac{t_{BA}}{L} \tag{12.3a}$$

Similarly, γ_B, the angle between the chord and the line tangent to the elastic curve at *B*, equals

$$\gamma_B = \frac{t_{AB}}{L} \tag{12.3b}$$

Since $\gamma_A = \theta_A - \psi_{AB}$ and $\gamma_B = \theta_B - \psi_{AB}$, we can express Equations 12.3*a* and 12.3*b* as

$$\theta_A - \psi_{AB} = \frac{t_{BA}}{L} \tag{12.4a}$$

$$\theta_B - \psi_{AB} = \frac{t_{AB}}{L} \tag{12.4b}$$

where

$$\psi_{AB} = \frac{\Delta_B - \Delta_A}{L} \tag{12.4c}$$

To express t_{AB} and t_{BA} in terms of the applied moments, we divide the ordinates of the moment curves in Figure 12.2*c* by *EI* to produce *M*/*EI* curves and, applying the second moment-area principle, sum the moments of the area under the *M*/*EI* curves about the *A* end of member *AB* to give t_{AB} and about the *B* end to give t_{BA}.

$$t_{AB} = \frac{M_{BA}}{EI}\frac{L}{2}\frac{2L}{3} - \frac{M_{AB}}{EI}\frac{L}{2}\frac{L}{3} - \frac{(A_M\bar{x})_A}{EI} \tag{12.5}$$

$$t_{BA} = \frac{M_{AB}}{EI}\frac{L}{2}\frac{2L}{3} - \frac{M_{BA}}{EI}\frac{L}{2}\frac{L}{3} + \frac{(A_M\bar{x})_B}{EI} \tag{12.6}$$

The first and second terms in Equations 12.5 and 12.6 represent the first moments of the triangular areas associated with the end moments M_{AB} and M_{BA}. The last term—$(A_M\bar{x})_A$ in Equation 12.5 and $(A_M\bar{x})_B$ in Equation

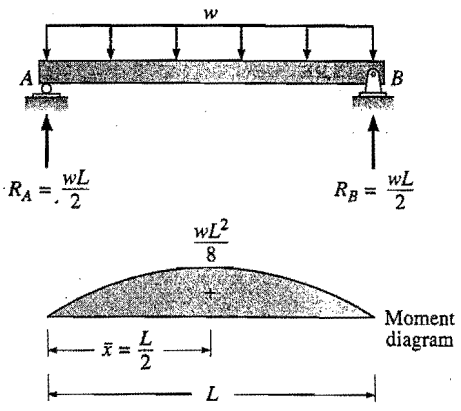

Figure 12.3: Simple beam moment curve produced by a uniform load.

12.6—represents the first moment of the area under the simple beam moment curve about the ends of the beam (the subscript indicates the end of the beam about which moments are taken). As a sign convention, we assume that the contribution of each moment curve to the tangential deviation is positive if it increases the tangential deviation and negative if it decreases the tangential deviation.

To illustrate the computation of $(A_M\bar{x})_A$ for a beam carrying a uniformly distributed load w (see Fig. 12.3), we draw the simple beam moment curve, a parabolic curve, and evaluate the product of the area under the curve and the distance $\bar{x}$ between point A and the centroid of the area:

$$(A_M\bar{x})_A = \text{area} \cdot \bar{x} = \frac{2L}{3}\frac{wL^2}{8}\left(\frac{L}{2}\right) = \frac{wL^4}{24} \tag{12.7}$$

Since the moment curve is symmetric, $(A_M\bar{x})_B$ equals $(A_M\bar{x})_A$.

If we next substitute the values of t_{AB} and t_{BA} given by Equations 12.5 and 12.6 into Equations 12.4*a* and 12.4*b*, we can write

$$\theta_A - \psi_{AB} = \frac{1}{L}\left[\frac{M_{BA}}{EI}\frac{L}{2}\frac{2L}{3} - \frac{M_{AB}}{EI}\frac{L}{2}\frac{L}{3} - \frac{(A_M\bar{x})_A}{EI}\right] \tag{12.8}$$

$$\theta_B - \psi_{AB} = \frac{1}{L}\left[\frac{M_{AB}}{EI}\frac{L}{2}\frac{2L}{3} - \frac{M_{BA}}{EI}\frac{L}{2}\frac{L}{3} - \frac{(A_M\bar{x})_B}{EI}\right] \tag{12.9}$$

To establish the slope-deflection equations, we solve Equations 12.8 and 12.9 simultaneously for M_{AB} and M_{BA} to give

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi_{AB}) + \frac{2(A_M\bar{x})_A}{L^2} - \frac{4(A_M\bar{x})_B}{L^2} \tag{12.10}$$

$$M_{BA} = \frac{2EI}{L}(2\theta_B + \theta_A - 3\psi_{AB}) + \frac{4(A_M\bar{x})_A}{L^2} - \frac{2(A_M\bar{x})_B}{L^2} \tag{12.11}$$

In Equations 12.10 and 12.11, the last two terms that contain the quantities $(A_M\bar{x})_A$ and $(A_M\bar{x})_B$ are a function of the loads applied between ends of the member only. We can give these terms a physical meaning by using Equations 12.10 and 12.11 to evaluate the moments in a fixed-end beam that has the same dimensions (cross section and span length) and supports the same load as member AB in Figure 12.2*a* (see Fig. 12.4). Since the ends of the beam in Figure 12.4 are fixed, the member end moments M_{AB} and M_{BA}, which are also termed *fixed-end moments*, may be designated FEM_{AB} and FEM_{BA}. Because the ends of the beam in Figure 12.4 are fixed against rotation and because no support settlements occur, it follows that

$$\theta_A = 0 \qquad \theta_B = 0 \qquad \psi_{AB} = 0$$

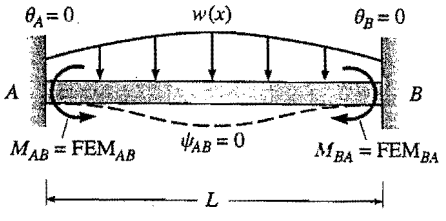

Figure 12.4

Substituting these values into Equations 12.10 and 12.11 to evaluate the member end moments (or fixed-end moments) in the beam of Figure 12.4, we can write

$$\text{FEM}_{AB} = M_{AB} = \frac{2(A_M\bar{x})_A}{L^2} - \frac{4(A_M\bar{x})_B}{L^2} \tag{12.12}$$

$$\text{FEM}_{BA} = M_{BA} = \frac{4(A_M\bar{x})_A}{L^2} - \frac{2(A_M\bar{x})_B}{L^2} \tag{12.13}$$

Using the results of Equations 12.12 and 12.13, we can write Equations 12.10 and 12.11 more simply by replacing the last two terms by FEM_{AB} and FEM_{BA} to produce

$$M_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B - 3\psi_{AB}) + \text{FEM}_{AB} \tag{12.14}$$

$$M_{BA} = \frac{2EI}{L}(2\theta_B + \theta_A - 3\psi_{AB}) + \text{FEM}_{BA} \tag{12.15}$$

Since Equations 12.14 and 12.15 have the same form, we can replace them with a single equation in which we denote the end where the moment is being computed as the near end (N) and the opposite end as the far end (F). With this adjustment we can write the slope-deflection equation as

$$M_{NF} = \frac{2EI}{L}(2\theta_N + \theta_F - 3\psi_{NF}) + \text{FEM}_{NF} \tag{12.16}$$

In Equation 12.16 the proportions of the member appear in the ratio I/L. This ratio, which is called the *relative flexural stiffness* of member NF, is denoted by the symbol K.

$$\text{Relative flexural stiffness } K = \frac{I}{L} \tag{12.17}$$

Substituting Equation 12.17 into Equation 12.16, we can write the slope-deflection equation as

$$M_{NF} = 2EK(2\theta_N + \theta_F - 3\psi_{NF}) + \text{FEM}_{NF} \tag{12.16a}$$

The value of the fixed-end moment (FEM_{NF}) in Equation 12.16 or 12.16*a* can be computed for any type of loading by Equations 12.12 and 12.13. The use of these equations to determine the fixed-end moments produced by a single concentrated load at midspan of a fixed-ended beam is illustrated in Example 12.1. See Figure 12.5. Values of fixed-end moments for other types of loading as well as support displacements are also given on the back cover.

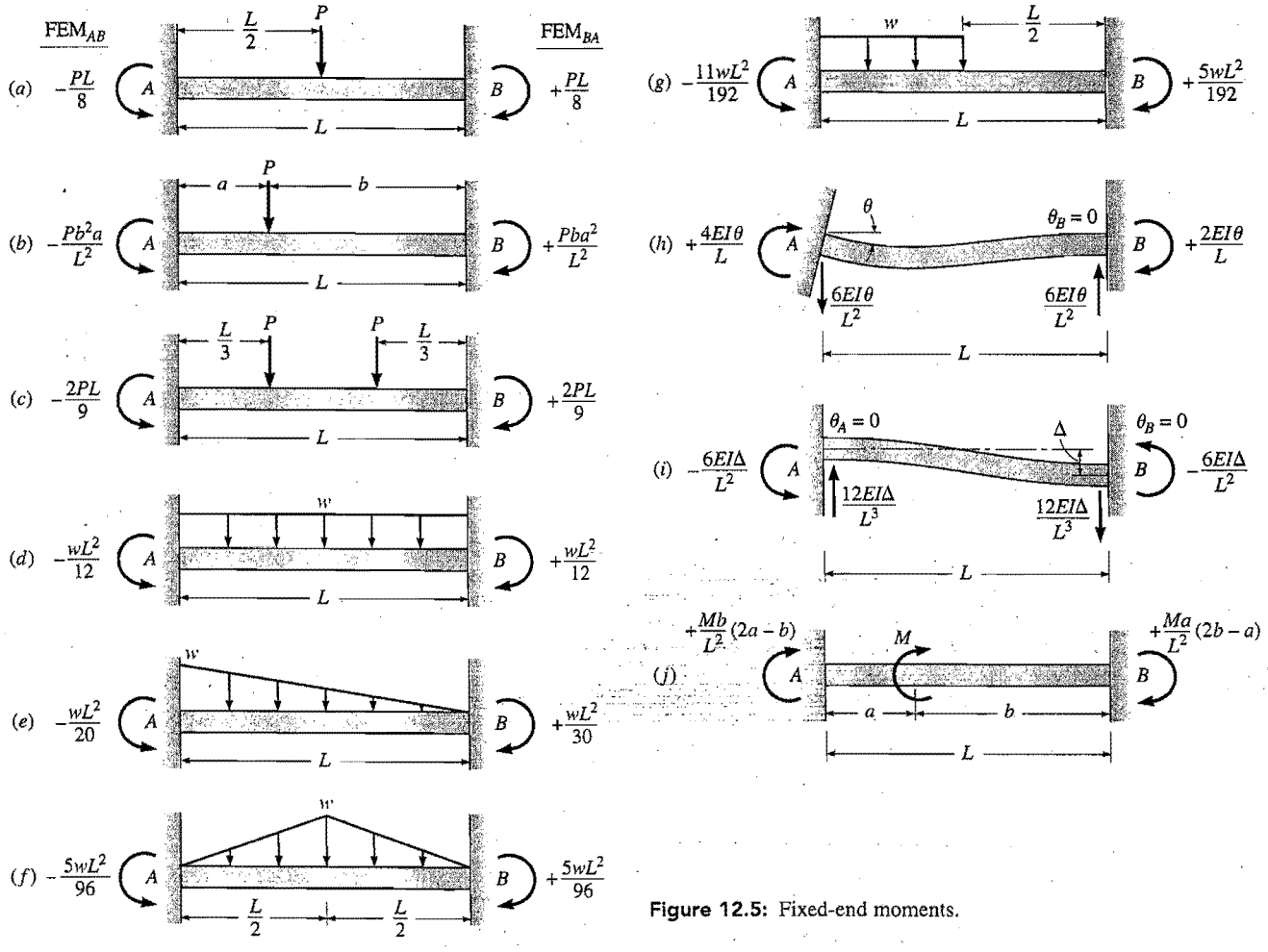

Figure 12.5: Fixed-end moments.

EXAMPLE 12.1

Using Equations 12.12 and 12.13, compute the fixed-end moments produced by a concentrated load P at midspan of the fixed-ended beam in Figure 12.6*a*. We know that EI is constant.

Solution

Equations 12.12 and 12.13 require that we compute, with respect to both ends of the beam in Figure 12.6*a*, the moment of the area under the simple beam moment curve produced by the applied load. To establish the simple beam moment curve, we imagine the beam AB in Figure 12.6*a* is removed from the fixed supports and placed on a set of simple supports, as shown in Figure 12.6*b*. The resulting simple beam moment curve pro-

duced by the concentrated load at midspan is shown in Figure 12.6*c*. Since the area under the moment curve is symmetric,

$$(A_M\bar{x})_A = (A_M\bar{x})_B = \frac{1}{2}L\frac{PL}{4}\left(\frac{L}{2}\right) = \frac{PL^3}{16}$$

Using Equation 12.12 yields

$$\text{FEM}_{AB} = \frac{2(A_M\bar{x})_A}{L^2} - \frac{4(A_M\bar{x})_B}{L^2}$$

$$= \frac{2}{L^2}\left(\frac{PL^3}{16}\right) - \frac{4}{L^2}\left(\frac{PL^3}{16}\right)$$

$$= -\frac{PL}{8} \quad \text{(the minus sign indicates a counterclockwise moment)} \quad \textbf{Ans.}$$

Using Equation 12.13 yields

$$\text{FEM}_{BA} = \frac{4(A_M\bar{x})_A}{L^2} - \frac{2(A_M\bar{x})_B}{L^2}$$

$$= \frac{4}{L^2}\left(\frac{PL^3}{16}\right) - \frac{2}{L^2}\left(\frac{PL^3}{16}\right) = +\frac{PL}{8} \quad \text{clockwise} \quad \textbf{Ans.}$$

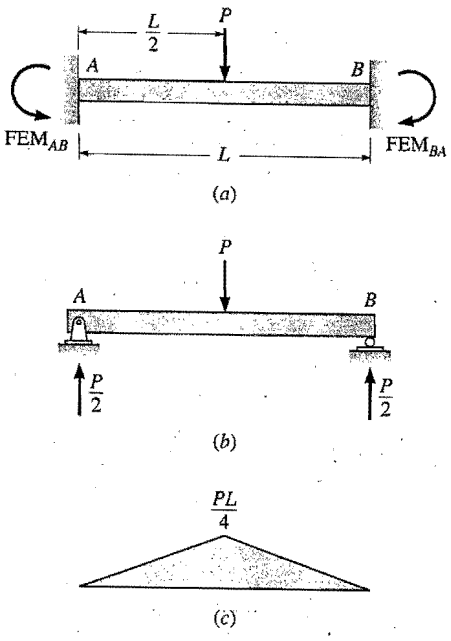

Figure 12.6

12.4 Analysis of Structures by the Slope-Deflection Method

Although the slope-deflection method can be used to analyze any type of indeterminate beam or frame, we will initially limit the method to indeterminate beams whose supports do not settle and to *braced* frames whose joints are free to rotate but are restrained against the displacement—restraint can be supplied by bracing members (Fig. 3.23*g*) or by supports. For these types of structures, the chord rotation angle ψ_{NF} in Equation 12.16 equals *zero*. Examples of several structures whose joints do not displace laterally but are free to rotate are shown in Figure 12.7*a* and *b*. In Figure 12.7*a* joint *A* is restrained against displacement by the fixed support and joint *C* by the pin support. Neglecting second-order changes in the length of members produced by bending and axial deformations, we can assume that joint *B* is restrained against horizontal displacement by member *BC*, which is connected to an immovable support at *C* and against vertical displacement by member *AB*, which connects to the fixed support at *A*. The approximate deflected shape of the loaded structures in Figure 12.7 is shown by dashed lines.

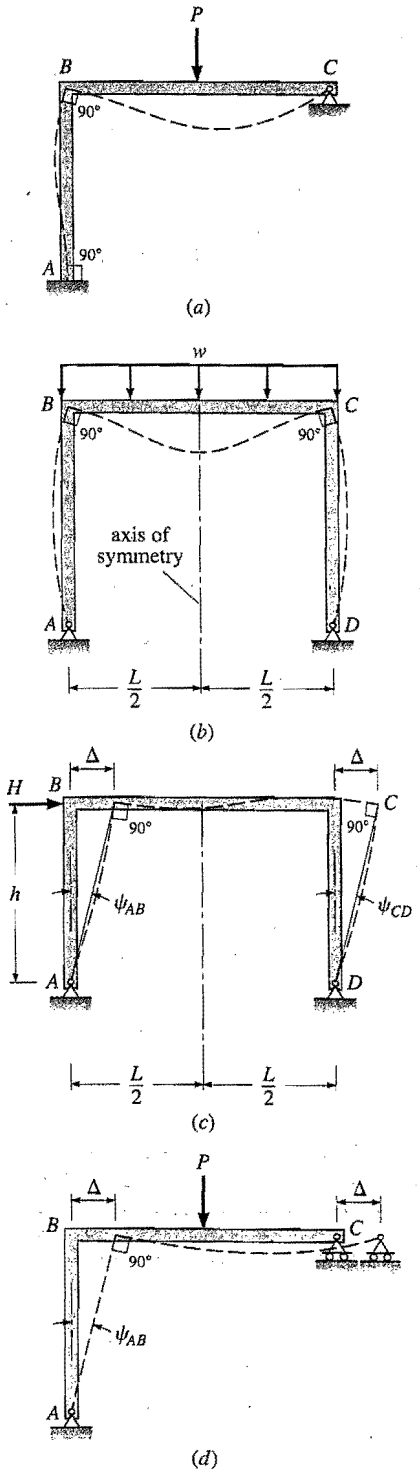

Figure 12.7*b* shows a structure whose configuration and loading are symmetric with respect to the vertical axis passing through the center of member *BC*. Since a symmetric structure under a symmetric load must deform in a symmetric pattern, no lateral displacement of the top joints can occur in either direction.

Figure 12.7*c* and *d* shows examples of frames that contain joints that are free to displace laterally as well as to rotate under the applied loads. Under the lateral load H, joints B and C in Figure 12.7*c* displace to the right. This displacement produces chord rotations $\psi = \Delta/h$ in members AB and CD. Since no vertical displacements of joints B and C occur—neglecting second-order bending and axial deformations of the columns—the chord rotation of the girder ψ_{BC} equals zero. Although the frame in Figure 12.7*d* supports a vertical load, joints B and C will displace laterally to the right a distance Δ because of the bending deformations of members AB and BC. We will consider the analysis of structures that contain one or more members with chord rotations in Section 12.5.

The basic steps of the slope-deflection method, which were discussed in Section 12.2, are summarized briefly below:

Summary

1. Identify all unknown joint displacements (rotations) to establish the number of unknowns.
2. Use the slope-deflection equation (Eq. 12.16) to express all member end moments in terms of joint rotations and the applied loads.
3. At each joint, except fixed supports, write the moment equilibrium equation, which states that the sum of the moments (applied by the members framing into the joint) equals zero. An equilibrium equation at a fixed support, which reduces to the identity $0 = 0$, supplies no useful information. The number of equilibrium equations must equal the number of unknown displacements.

   As a sign convention, *clockwise moments on the ends of the members are assumed to be positive.* If a moment at the end of a member is unknown, it must be shown clockwise on the end of a member. The moment applied by a member to a joint is always equal and opposite in direction to the moment acting on the end of the member. If the magnitude and direction of the moment on the end of a member are known, they are shown in the actual direction.
4. Substitute the expressions for moments as a function of displacements (see step 2) into the equilibrium equations in step 3, and solve for the unknown displacements.

Figure 12.7: (*a*) All joints restrained against displacement; all chord rotations ψ equal zero; (*b*) due to symmetry of structure and loading, joints free to rotate but not translate; chord rotations equal zero; (*c*) and (*d*) unbraced frames with chord rotations.

5. Substitute the values of displacement in step 4 into the expressions for member end moment in step 2 to establish the value of the member end moments. Once the member end moments are known, the balance of the analysis—drawing shear and moment curves or computing reactions, for example—is completed by statics.

Examples 12.2 and 12.3 illustrate the procedure outlined above.

EXAMPLE 12.2

Using the slope-deflection method, determine the member end moments in the indeterminate beam shown in Figure 12.8*a*. The beam, which behaves elastically, carries a concentrated load at midspan. After the end moments are determined, draw the shear and moment curves. If $I = 240$ in^4 and $E = 30{,}000$ kips/in^2, compute the magnitude of the slope at joint B.

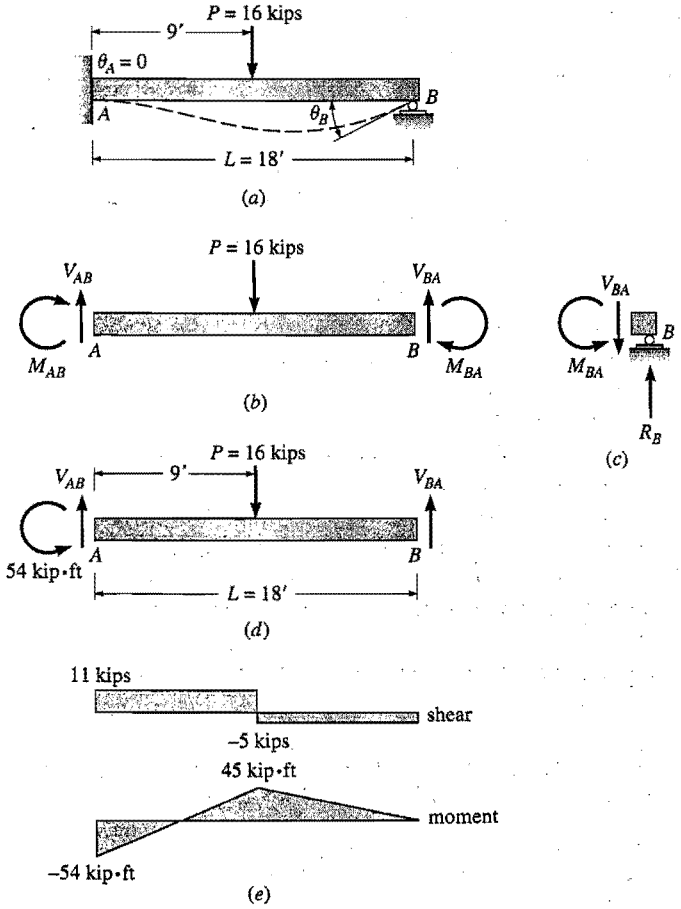

Figure 12.8: (*a*) Beam with one unknown displacement θ_B; (*b*) free body of beam AB; unknown member end moments M_{AB} and M_{BA} shown clockwise; (*c*) free body of joint B; (*d*) free body used to compute end shears; (*e*) shear and moment curves.

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Example 12.2 continues . . .

Solution

Since joint A is fixed against rotation, $\theta_A = 0$; therefore, the only unknown displacement is θ_B, the rotation of joint B (ψ_{AB} is, of course, zero since no support settlements occur). Using the slope-deflection equation

$$M_{NF} = \frac{2EI}{L}(2\theta_N + \theta_F - 3\psi_{NF}) + \text{FEM}_{NF} \qquad (12.16)$$

and the values in Figure 12.5*a* for the fixed-end moments produced by a concentrated load at midspan, we can express the member end moments shown in Figure 12.8*b* as

$$M_{AB} = \frac{2EI}{L}(\theta_B) - \frac{PL}{8} \qquad (1)$$

$$M_{BA} = \frac{2EI}{L}(2\theta_B) + \frac{PL}{8} \qquad (2)$$

To determine θ_B, we next write the equation of moment equilibrium at joint B (see Fig. 12.8*c*):

$$\circlearrowleft+ \quad \Sigma M_B = 0$$

$$M_{BA} = 0 \qquad (3)$$

Substituting the value of M_{BA} given by Equation 2 into Equation 3 and solving for θ_B give

$$\frac{4EI}{L}\theta_B + \frac{PL}{8} = 0$$

$$\theta_B = -\frac{PL^2}{32EI} \qquad (4)$$

where the minus sign indicates both that the B end of member AB and joint B rotate in the counterclockwise direction. To determine the member end moments, the value of θ_B given by Equation 4 is substituted into Equations 1 and 2 to give

$$M_{AB} = \frac{2EI}{L}\left(\frac{-PL^2}{32EI}\right) - \frac{PL}{8} = -\frac{3PL}{16} = -54 \text{ kip·ft} \qquad \textbf{Ans.}$$

$$M_{BA} = \frac{4EI}{L}\left(\frac{-PL^2}{32EI}\right) + \frac{PL}{8} = 0$$

Although we know that M_{BA} is zero since the support at B is a pin, the computation of M_{BA} serves as a check.

To complete the analysis, we apply the equations of statics to a free body of member AB (see Fig. 12.8*d*).

$$\circlearrowleft^{+} \quad \Sigma M_A = 0$$

$$0 = (16 \text{ kips})(9 \text{ ft}) - V_{BA}(18 \text{ ft}) - 54 \text{ kip·ft}$$

$$V_{BA} = 5 \text{ kips}$$

$$\overset{+}{\uparrow} \quad \Sigma F_y = 0$$

$$0 = V_{BA} + V_{AB} - 16$$

$$V_{AB} = 11 \text{ kips}$$

To evaluate θ_B, we express all variables in Equation 4 in units of inches and kips.

$$\theta_B = -\frac{PL^2}{32EI} = -\frac{16(18 \times 12)^2}{32(30{,}000)240} = -0.0032 \text{ rad}$$

Expressing θ_B in degrees, we obtain

$$\frac{2\pi \text{ rad}}{360^\circ} = \frac{-0.0032}{\theta_B}$$

$$\theta_B = -0.183^\circ \qquad \textbf{Ans.}$$

Note that the slope θ_B is extremely small and not discernible to the naked eye.

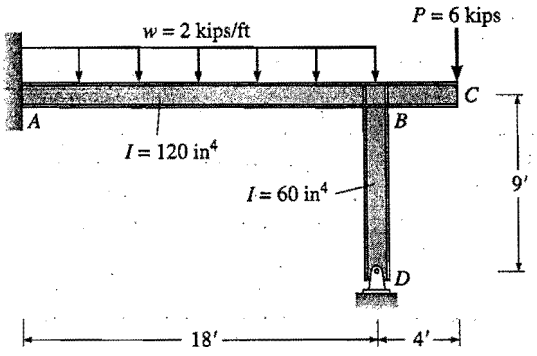
NOTE. When you analyze a structure by the slope-deflection method, you must follow a rigid format in formulating the equilibrium equations. There is no need to guess the direction of unknown member end moments since the solution of the equilibrium equations will automatically produce the correct direction for displacements and moments. For example, in Figure 12.8*b* we show the moments M_{AB} and M_{BA} clockwise on the ends of member *AB* even though intuitively we may recognize from a sketch of the deflected shape in Figure 12.8*a* that moment M_{AB} must act in the counterclockwise direction because the beam is bent concave downward at the left end by the load. When the solution indicates M_{AB} is -54 kip·ft, we know from the negative sign that M_{AB} actually acts on the end of the member in the counterclockwise direction.

EXAMPLE 12.3

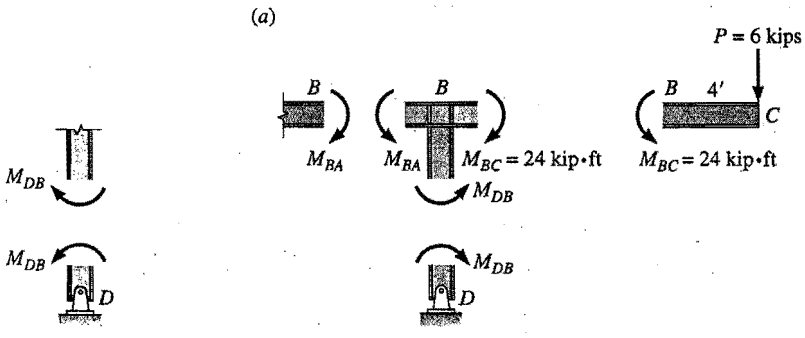
Using the slope-deflection method, determine the member end moments in the braced frame shown in Figure 12.9*a*. Also compute the reactions at support *D*, and draw the shear and moment curves for members *AB* and *BD*.

[*continues on next page*]

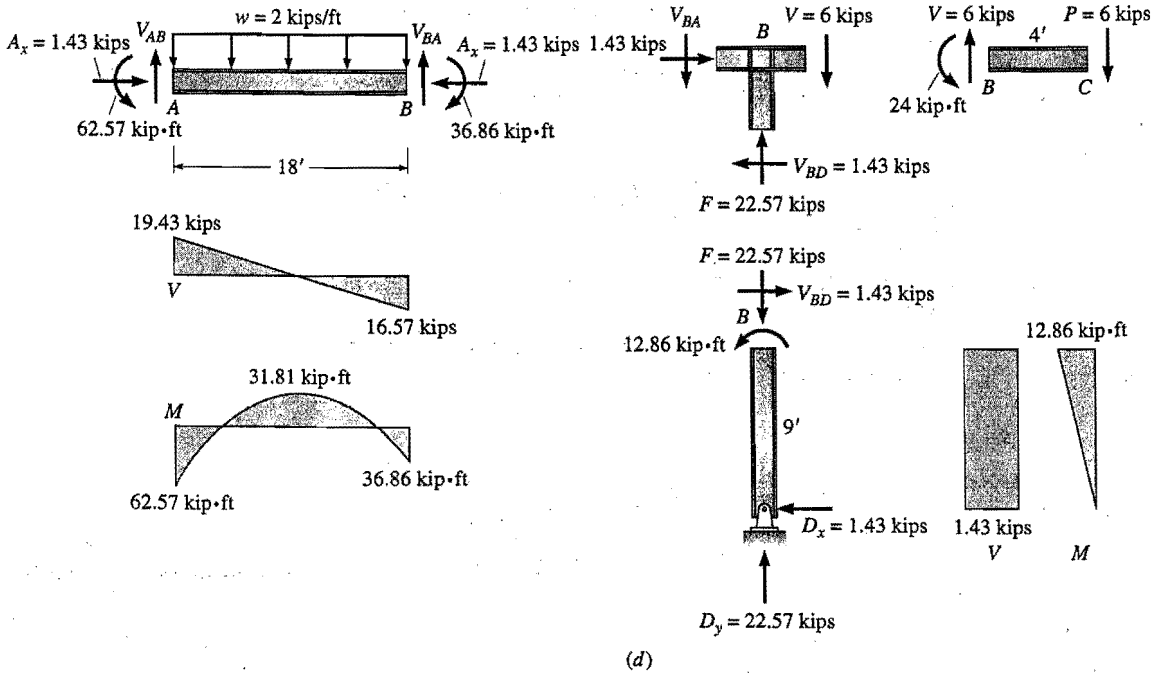
Example 12.3 continues . . .



(a)



(b) (c)



(d)

Figure 12.9: (*a*) Frame details; (*b*) joint *D*; (*c*) joint *B* (shears and axial forces omitted for clarity); (*d*) free bodies of members and joints used to compute shears and reactions (moments acting on joint *B* omitted for clarity).

Solution

Since θ_A equals zero because of the fixed support at A, θ_B and θ_D are the only unknown joint displacements we must consider. Although the moment applied to joint B by the cantilever BC must be included in the joint equilibrium equation, there is no need to include the cantilever in the slope-deflection analysis of the indeterminate portions of the frame because the cantilever is determinate; that is, the shear and the moment at any section of member BC can be determined by the equations of statics. In the slope-deflection solution, we can treat the cantilever as a device that applies a vertical force of 6 kips and a clockwise moment of 24 kip·ft to joint B.

Using the slope-deflection equation

$$M_{NF} = \frac{2EI}{L}(2\theta_N + \theta_F - 3\psi_{NF}) + \text{FEM}_{NF} \quad (12.16)$$

where all variables are expressed in units of kip·inches and the fixed-end moments produced by the uniform load on member AB (see Fig. 12.5d) equal

$$\text{FEM}_{AB} = -\frac{wL^2}{12}$$

$$\text{FEM}_{BA} = +\frac{wL^2}{12}$$

we can express the member end moments as

$$M_{AB} = \frac{2E(120)}{18(12)}(\theta_B) - \frac{2(18)^2(12)}{12} = 1.11E\theta_B - 648 \quad (1)$$

$$M_{BA} = \frac{2E(120)}{18(12)}(2\theta_B) + \frac{2(18)^2(12)}{12} = 2.22E\theta_B + 648 \quad (2)$$

$$M_{BD} = \frac{2E(60)}{9(12)}(2\theta_B + \theta_D) = 2.22E\theta_B + 1.11E\theta_D \quad (3)$$

$$M_{DB} = \frac{2E(60)}{9(12)}(2\theta_D + \theta_B) = 2.22E\theta_D + 1.11E\theta_B \quad (4)$$

To solve for the unknown joint displacements θ_B and θ_D, we write equilibrium equations at joints D and B.

At joint D (see Fig. 12.9b): $+\circlearrowleft \quad \Sigma M_D = 0$

$$M_{DB} = 0 \quad (5)$$

At joint B (see Fig. 12.9c): $+\circlearrowleft \quad \Sigma M_B = 0$

$$M_{BA} + M_{BD} - 24(12) = 0 \quad (6)$$

[continues on next page]

Example 12.3 continues . . .

Since the magnitude and direction of the moment M_{BC} at the B end of the cantilever can be evaluated by statics (summing moments about point B), it is applied in the correct sense (counterclockwise) on the end of member BC, as shown in Figure 12.9*c*. On the other hand, since the magnitude and direction of the end moments M_{BA} and M_{BD} are unknown, they are assumed to act in the positive sense—clockwise on the ends of the members and counterclockwise on the joint.

Using Equations 2 to 4 to express the moments in Equations 5 and 6 in terms of displacements, we can write the equilibrium equations as

At joint D: $$2.22E\theta_D + 1.11E\theta_B = 0 \quad (7)$$

At joint B: $$(2.22E\theta_B + 648) + (2.22E\theta_B + 1.11E\theta_D) - 288 = 0 \quad (8)$$

Solving Equations 7 and 8 simultaneously gives

$$\theta_D = \frac{46.33}{E}$$

$$\theta_B = -\frac{92.66}{E}$$

To establish the values of the member end moments, the values of θ_B and θ_D above are substituted into Equations 1, 2, and 3, giving

$$M_{AB} = 1.11E\left(-\frac{92.66}{E}\right) - 648$$

$$= -750.85 \text{ kip·in} = -62.57 \text{ kip·ft} \qquad \textbf{Ans.}$$

$$M_{BA} = 2.22E\left(-\frac{92.66}{E}\right) + 648$$

$$= 442.29 \text{ kip·in} = +36.86 \text{ kip·ft} \qquad \textbf{Ans.}$$

$$M_{BD} = 2.22E\left(-\frac{92.66}{E}\right) + 1.11E\left(\frac{46.33}{E}\right)$$

$$= -154.28 \text{ kip·in} = -12.86 \text{ kip·ft} \qquad \textbf{Ans.}$$

Now that the member end moments are known, we complete the analysis by using the equations of statics to determine the shears at the ends of all members. Figure 12.9*d* shows free-body diagrams of both members and joints: Except for the cantilever, all members carry axial forces as well as shear and moment. After the shears are computed, axial forces and reactions can be evaluated by considering the equilibrium of the joints. For example, vertical equilibrium of the forces applied to joint B requires that the vertical force F in column BD equal the sum of the shears applied to joint B by the B ends of members AB and BC.

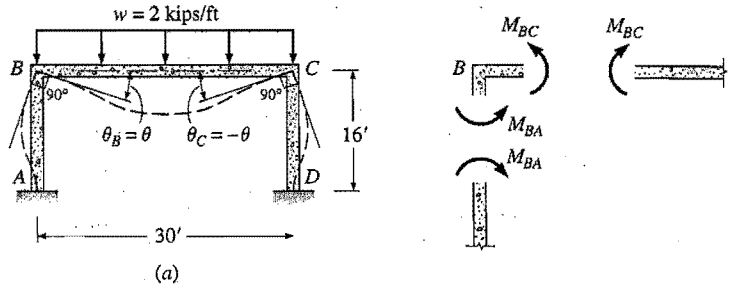
EXAMPLE 12.4

Use of Symmetry to Simplify the Analysis of a Symmetric Structure with a Symmetric Load

Determine the reactions and draw the shear and moment curves for the columns and girder of the rigid frame shown in Figure 12.10*a*. Given: $I_{AB} = I_{CD} = 120$ in^4, $I_{BC} = 360$ in^4, and E is constant for all members.

Solution

Although joints B and C rotate, they do not displace laterally because both the structure and its load are symmetric with respect to a vertical axis of symmetry passing through the center of the girder. Moreover, θ_B and θ_C are equal in magnitude; however, θ_B, a clockwise rotation, is positive,



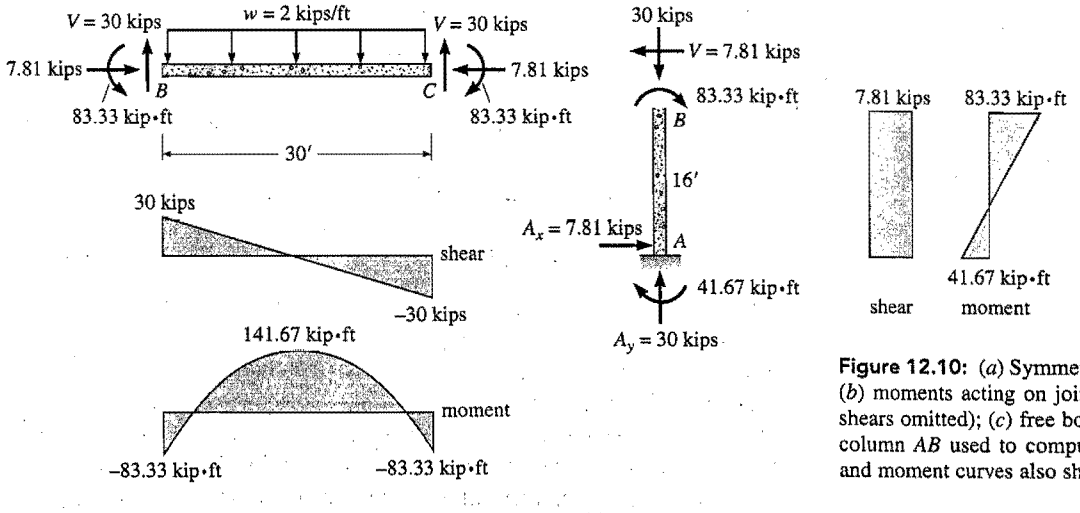


Figure 12.10: (*a*) Symmetric structure and load; (*b*) moments acting on joint B (axial forces and shears omitted); (*c*) free bodies of girder BC and column AB used to compute shears; final shear and moment curves also shown.

[*continues on next page*]

Example 12.4 continues . . .

and θ_C, a counterclockwise rotation, is negative. Since the problem contains only one unknown joint rotation, we can determine its magnitude by writing the equilibrium equation for either joint B or joint C. We will arbitrarily choose joint B.

Expressing member end moments with Equation 12.16, reading the value of fixed-end moment for member BC from Figure 12.5*d*, expressing units in kips·inch, and substituting $\theta_B = \theta$ and $\theta_C = -\theta$, we can write

$$M_{AB} = \frac{2E(120)}{16(12)}(\theta_B) = 1.25E\theta_B \tag{1}$$

$$M_{BA} = \frac{2E(120)}{16(12)}(2\theta_B) = 2.50E\theta_B \tag{2}$$

$$M_{BC} = \frac{2E(360)}{30(12)}(2\theta_B + \theta_C) - \frac{wL^2}{12}$$

$$= 2E[2\theta + (-\theta)] - \frac{2(30)^2(12)}{12} = 2E\theta - 1800 \tag{3}$$

Writing the equilibrium equation at joint B (see Fig. 12.10*b*) yields

$$M_{BA} + M_{BC} = 0 \tag{4}$$

Substituting Equations 2 and 3 into Equation 4 and solving for θ produce

$$2.5E\theta + 2.0E\theta - 1800 = 0$$

$$\theta = \frac{400}{E} \tag{5}$$

Substituting the value of θ given by Equation 5 into Equations 1, 2, and 3 gives

$$M_{AB} = 1.25E\left(\frac{400}{E}\right)$$

$$= 500 \text{ kip·in} = 41.67 \text{ kip·ft} \quad \textbf{Ans.}$$

$$M_{BA} = 2.5E\left(\frac{400}{E}\right)$$

$$= 1000 \text{ kip·in} = 83.33 \text{ kip·ft} \quad \textbf{Ans.}$$

$$M_{BC} = 2E\left(\frac{400}{E}\right) - 1800$$

$$= -1000 \text{ kip·in} = -83.33 \text{ kip·ft counterclockwise} \quad \textbf{Ans.}$$

The final results of the analysis are shown in Figure 12.10*c*.

EXAMPLE 12.5

Using symmetry to simplify the slope-deflection analysis of the frame in Figure 12.11*a*, determine the reactions at supports *A* and *D*.

Solution

Examination of the frame shows that all joint rotations are zero. Both θ_A and θ_C are zero because of the fixed supports at *A* and *C*. Since column *BD* lies on the vertical axis of symmetry, we can infer that it must remain straight since the deflected shape of the structure with respect to the axis of symmetry must be symmetric. If the column were to bend in either direction, the requirement that the pattern of deformations be symmetric

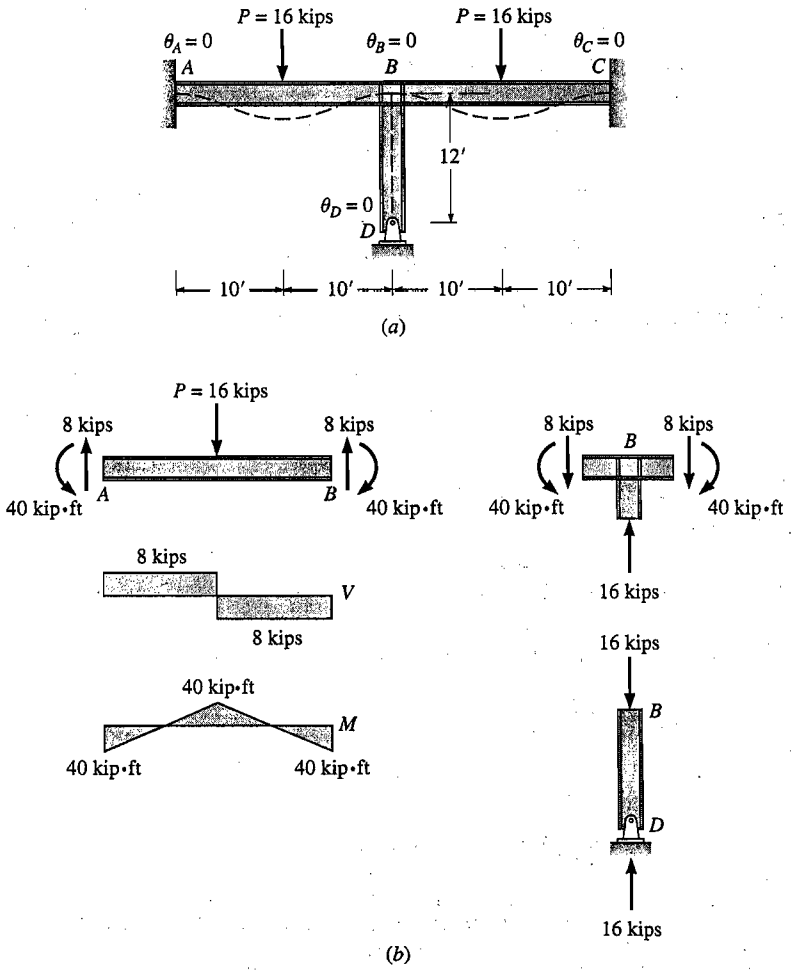


Figure 12.11: (*a*) Symmetric frame with symmetric load (deflected shape shown by dashed line); (*b*) free body of beam *AB*, joint *B*, and column *BD*. Final shear and moment diagrams for beam *AB*.

[*continues on next page*]

Example 12.5 continues . . .

would be violated. Since the column remains straight, neither the top nor bottom joints at B and D rotate; therefore, both θ_B and θ_D equal zero. Because no support settlements occur, chord rotations for all members are zero. Since all joint and chord rotations are zero, we can see from the slope-deflection equation (Eq. 12.16) that the member end moments at each end of beams AB and BC are equal to the fixed-end moments $PL/8$ given by Figure 12.5*a*:

$$\text{FEM} = \pm\frac{PL}{8} = \frac{16(20)}{8} = \pm 40 \text{ kip·ft}$$

Free bodies of beam AB, joint B, and column BD are shown in Figure 12.11.

NOTE. The analysis of the frame in Figure 12.11 shows that column BD carries only axial load because the moments applied by the beams to each side of the joint are the same. A similar condition often exists at the interior columns of multistory buildings whose structure consists of either a continuous reinforced concrete or a welded-steel rigid-jointed frame. Although a rigid joint has the capacity to transfer moments from the beams to the column, it is the *difference* between the moments applied by the girders on either side of a joint that determines the moment to be transferred. When the span lengths of the beams and the loads they support are approximately the same (a condition that exists in most buildings), the difference in moment is small. As a result, in the preliminary design stage most columns can be sized accurately by considering only the magnitude of the axial load produced by the gravity load from the tributary area supported by the column.

EXAMPLE 12.6

Determine the reactions and draw the shear and moment curves for the beam in Figure 12.12. The support at A has been accidentally constructed with a slope that makes an angle of 0.009 rad with the vertical y-axis through support A, and B has been constructed 1.2 in below its intended position. Given: EI is constant, $I = 360$ in^4, and $E = 29{,}000$ kips/in^2.

Solution

The slope at A and the chord rotation ψ_{AB} can be determined from the information supplied about the support displacements. Since the end of the beam is rigidly connected to the fixed support at A, it rotates counterclockwise with the support; and $\theta_A = -0.009$ rad. The settlement of support B relative to support A produces a clockwise chord rotation

$$\psi_{AB} = \frac{\Delta}{L} = \frac{1.2}{20(12)} = 0.005 \text{ radians}$$

Angle θ_B is the only unknown displacement, and the fixed-end moments are zero because no loads act on beam. Expressing member end moments with the slope-deflection equation (Eq. 12.16), we have

$$M_{AB} = \frac{2EI_{AB}}{L_{AB}}(2\theta_A + \theta_B - 3\psi_{AB}) + \text{FEM}_{AB}$$

$$M_{AB} = \frac{2E(360)}{20(12)}[2(-0.009) + \theta_B - 3(0.005)] \qquad (1)$$

$$M_{BA} = \frac{2E(360)}{20(12)}[2\theta_B + (-0.009) - 3(0.005)] \qquad (2)$$

Writing the equilibrium equation at joint B yields

$$+\circlearrowleft \quad \Sigma M_B = 0$$

$$M_{BA} = 0 \qquad (3)$$

Substituting Equation 2 into Equation 3 and solving for θ_B yield

$$3E(2\theta_B - 0.009 - 0.015) = 0$$

$$\theta_B = 0.012 \text{ radians}$$

To evaluate M_{AB}, substitute θ_B into Equation 1:

$$M_{AB} = 3(29{,}000)[2(-0.009) + 0.012 - 3(0.005)]$$

$$= -1827 \text{ kip·in} = -152.25 \text{ kip·ft}$$

Complete the analysis by using the equations of statics to compute the reaction at B and the shear at A (see Fig. 12.12*b*).

$$\circlearrowleft^+ \quad \Sigma M_A = 0$$

$$0 = R_B(20) - 152.25$$

$$R_B = 7.61 \text{ kips} \qquad \textbf{Ans.}$$

$$+\uparrow \quad \Sigma F_y = 0$$

$$V_A = 7.61 \text{ kips}$$

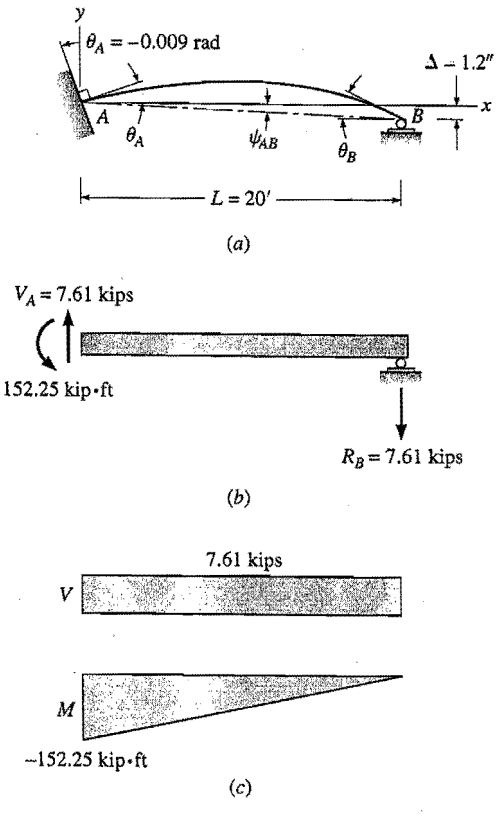


Figure 12.12: (*a*) Deformed shape; (*b*) free body used to compute V_A and R_B; (*c*) shear and moment curves.

EXAMPLE 12.7

Although the supports are constructed in their correct position, girder AB of the frame shown in Figure 12.13 is fabricated 1.2 in too long. Determine the reactions created when the frame is connected into the supports. Given: EI is a constant for all members, $I = 240$ in^4, and $E = 29{,}000$ kips/in^2.

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Example 12.7 continues . . .

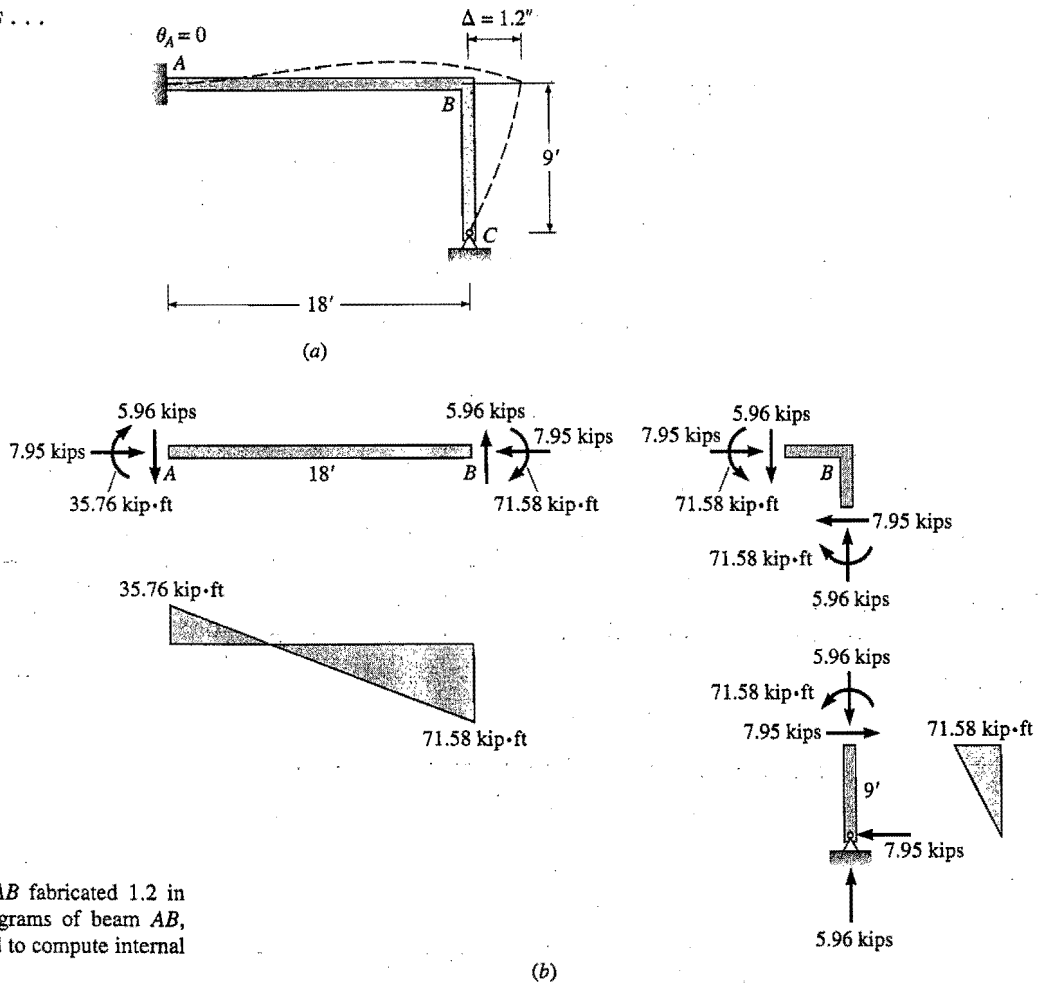


Figure 12.13: (*a*) Girder *AB* fabricated 1.2 in too long; (*b*) free-body diagrams of beam *AB*, joint *B*, and column *BC* used to compute internal forces and reactions.

Solution

The deflected shape of the frame is shown by the dashed line in Figure 12.13*a*. Although internal forces (axial, shear, and moment) are created when the frame is forced into the supports, the deformations produced by these forces are neglected since they are small compared to the 1.2-in fabrication error; therefore, the chord rotation ψ_{BC} of column *BC* equals

$$\psi_{BC} = \frac{\Delta}{L} = \frac{1.2}{9(12)} = \frac{1}{90} \text{ rad}$$

Since the ends of girder *AB* are at the same level, $\psi_{AB} = 0$. The unknown displacements are θ_B and θ_C.

Using the slope-deflection equation (Eq. 12.16), we express member end moments in terms of the unknown displacements. Because no loads are applied to the members, all fixed-end moments equal zero.

$$M_{AB} = \frac{2E(240)}{18(12)}(\theta_B) = 2.222E\theta_B \quad (1)$$

$$M_{BA} = \frac{2E(240)}{18(12)}(2\theta_B) = 4.444E\theta_B \quad (2)$$

$$M_{BC} = \frac{2E(240)}{9(12)}\left[2\theta_B + \theta_C - 3\left(\frac{1}{90}\right)\right]$$

$$= 8.889E\theta_B + 4.444E\theta_C - 0.1481E \quad (3)$$

$$M_{CB} = \frac{2E(240)}{9(12)}\left[2\theta_C + \theta_B - 3\left(\frac{1}{90}\right)\right]$$

$$= 8.889E\theta_C + 4.444E\theta_B - 0.1481E \quad (4)$$

Writing equilibrium equations gives

Joint C: $$M_{CB} = 0 \quad (5)$$

Joint B: $$M_{BA} + M_{BC} = 0 \quad (6)$$

Substituting Equations 2 to 4 into Equations 5 and 6 solving for θ_B and θ_C yield

$$8.889E\theta_C + 4.444E\theta_B - 0.1481E = 0$$

$$4.444E\theta_B + 8.889E\theta_B + 4.444E\theta_C - 0.1481E = 0$$

$$\theta_B = 0.00666 \text{ rad} \quad (7)$$

$$\theta_C = 0.01332 \text{ rad} \quad (8)$$

Substituting θ_C and θ_B into Equations 1 to 3 produces

$$M_{AB} = 35.76 \text{ kip·ft} \qquad M_{BA} = 71.58 \text{ kip·ft} \qquad \textbf{Ans.}$$

$$M_{BC} = -71.58 \text{ kip·ft} \qquad M_{CB} = 0$$

The free-body diagrams used to compute internal forces and reactions are shown in Figure 12.13*b*, which also shows moment diagrams.

12.5 Analysis of Structures That Are Free to Sidesway

Thus far we have used the slope-deflection method to analyze indeterminate beams and frames with joints that are free to rotate but which are restrained against displacement. We now extend the method to frames

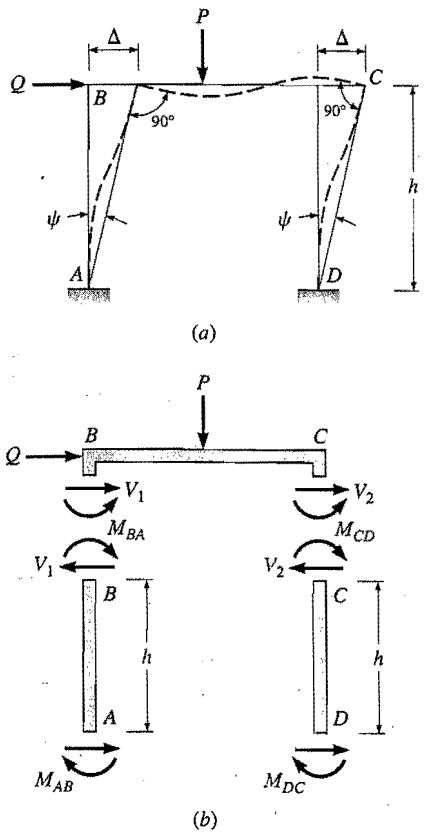


Figure 12.14: (*a*) Unbraced frame, deflected shape shown to an exaggerated scale by dashed lines, column chords rotate through a clockwise angle ψ; (*b*) free-body diagrams of columns and girders; unknown moments shown in the positive sense, that is, clockwise on ends of members (axial loads in columns and shears in girder omitted for clarity).

whose joints are also free to *sidesway*, that is, to displace laterally. For example, in Figure 12.14*a* the horizontal load results in girder *BC* displacing laterally a distance Δ. Recognizing that the axial deformation of the girder is insignificant, we assume that the horizontal displacement of the top of both columns equals Δ. This displacement creates a clockwise chord rotation ψ in both legs of the frame equal to

$$\psi = \frac{\Delta}{h}$$

where h is the length of column.

Since three independent displacements develop in the frame [i.e., the rotation of joints B and C (θ_B and θ_C) and the chord rotation ψ], we require three equilibrium equations for their solution. Two equilibrium equations are supplied by considering the equilibrium of the moments acting on joints B and C. Since we have written equations of this type in the solution of previous problems, we will only discuss the second type of equilibrium equation—the *shear equation*. The shear equation is established by summing in the horizontal direction the forces acting on a free body of the girder. For example, for the girder in Figure 12.14*b* we can write

$$\rightarrow+ \quad \Sigma F_x = 0$$

$$V_1 + V_2 + Q = 0 \tag{12.18}$$

In Equation 12.18, V_1, the shear in column *AB*, and V_2, the shear in column *CD*, are evaluated by summing moments about the bottom of each column of the forces acting on a free body of the column. As we established previously, the unknown moments on the ends of the column must always be shown in the positive sense, that is, acting clockwise on the end of the member. Summing moments about point *A* of column *AB*, we compute V_1:

$$\circlearrowright^+ \quad \Sigma M_A = 0$$

$$M_{AB} + M_{BA} - V_1 h = 0$$

$$V_1 = \frac{M_{AB} + M_{BA}}{h} \tag{12.19}$$

Similarly, the shear in column *CD* is evaluated by summing moments about point *D*.

$$\circlearrowright^+ \quad \Sigma M_D = 0$$

$$M_{CD} + M_{DC} - V_2 h = 0$$

$$V_2 = \frac{M_{CD} + M_{DC}}{h} \tag{12.20}$$

Substituting the values of V_1 and V_2 from Equations 12.19 and 12.20 into Equation 12.18, we can write the third equilibrium equation as

$$\frac{M_{AB} + M_{BA}}{h} + \frac{M_{CD} + M_{DC}}{h} + Q = 0 \qquad (12.21)$$

Examples 12.8 and 12.9 illustrate the use of the slope-deflection method to analyze frames that carry lateral loads and are free to sidesway. Frames that carry only vertical load will also undergo small amounts of sidesway unless both the structure and the loading pattern are symmetric. Example 12.10 illustrates this case.

EXAMPLE 12.8

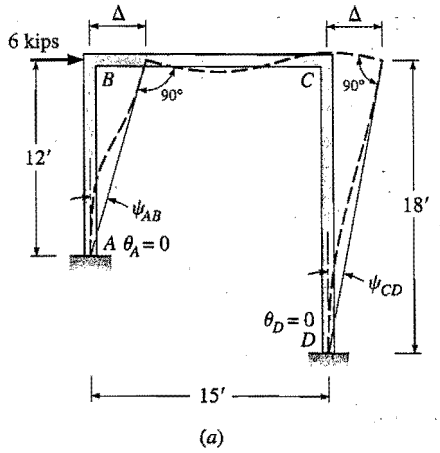
Analyze the frame in Figure 12.15*a* by the slope-deflection method. E is constant for all members.

$$I_{AB} = 240 \text{ in}^4 \qquad I_{BC} = 600 \text{ in}^4 \qquad I_{CD} = 360 \text{ in}^4$$

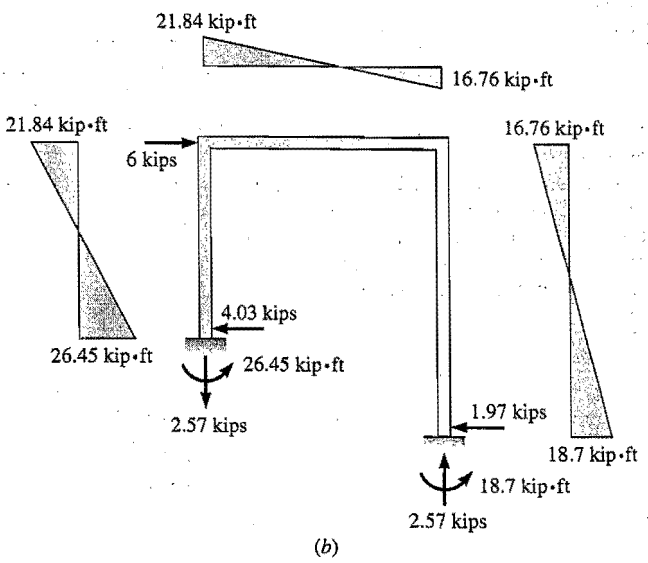
Solution

Identify the unknown displacements θ_B, θ_C, and Δ. Express the chord rotations ψ_{AB} and ψ_{CD} in terms of Δ:

$$\psi_{AB} = \frac{\Delta}{12} \quad \text{and} \quad \psi_{CD} = \frac{\Delta}{18} \quad \text{so} \quad \psi_{AB} = 1.5\psi_{CD} \qquad (1)$$



(*a*)



(*b*)

Figure 12.15: (*a*) Details of frame; (*b*) reactions and moment diagrams.

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Example 12.8 continues . . .

Compute the relative bending stiffness of all members.

$$K_{AB} = \frac{EI}{L} = \frac{240E}{12} = 20E$$

$$K_{BC} = \frac{EI}{L} = \frac{600E}{15} = 40E$$

$$K_{CD} = \frac{EI}{L} = \frac{360E}{18} = 20E$$

If we set $20E = K$, then

$$K_{AB} = K \qquad K_{BC} = 2K \qquad K_{CD} = K \tag{2}$$

Express member end moments in terms of displacements with slope-deflection equation 12.16: $M_{NF} = (2EI/L)(2\theta_N + \theta_F - 3\psi_{NF}) + \text{FEM}_{NF}$. Since no loads are applied to members between joints, all $\text{FEM}_{NF} = 0$.

$$\begin{aligned} M_{AB} &= 2K_{AB}(\theta_B - 3\psi_{AB}) \\ M_{BA} &= 2K_{AB}(2\theta_B - 3\psi_{AB}) \\ M_{BC} &= 2K_{BC}(2\theta_B + \theta_C) \\ M_{CB} &= 2K_{BC}(2\theta_C + \theta_B) \\ M_{CD} &= 2K_{CD}(2\theta_C - 3\psi_{CD}) \\ M_{DC} &= 2K_{CD}(\theta_C - 3\psi_{CD}) \end{aligned} \tag{3}$$

In the equations above, use Equations 1 to express ψ_{AB} in terms of ψ_{CD}, and use Equations 2 to express all stiffness in terms of the parameter K.

$$\begin{aligned} M_{AB} &= 2K(\theta_B - 4.5\psi_{CD}) \\ M_{BA} &= 2K(2\theta_B - 4.5\psi_{CD}) \\ M_{BC} &= 4K(2\theta_B + \theta_C) \\ M_{CB} &= 4K(2\theta_C + \theta_B) \\ M_{CD} &= 2K(2\theta_C - 3\psi_{CD}) \\ M_{DC} &= 2K(\theta_C - 3\psi_{CD}) \end{aligned} \tag{4}$$

The equilibrium equations are:

Joint B: $$M_{BA} + M_{BC} = 0 \tag{5}$$

Joint C: $$M_{CB} + M_{CD} = 0 \tag{6}$$

Shear equation (see Eq. 12.21): $$\frac{M_{BA} + M_{AB}}{12} + \frac{M_{CD} + M_{DC}}{18} + 6 = 0 \tag{7}$$

Substitute Equations 4 into Equations 5, 6, and 7 and combine terms.

$$12\theta_B + 4\theta_C - 9\psi_{CD} = 0 \qquad (5a)$$

$$4\theta_B + 12\theta_C - 6\psi_{CD} = 0 \qquad (6a)$$

$$9\theta_B + 6\theta_C - 39\psi_{CD} = -\frac{108}{K} \qquad (7a)$$

Solving the equations above simultaneously gives

$$\theta_B = \frac{2.257}{K} \qquad \theta_C = \frac{0.97}{K} \qquad \psi_{CD} = \frac{3.44}{K}$$

Also, $$\psi_{AB} = 1.5\psi_{CD} = \frac{5.16}{K}$$

Since all angles are positive, all joint rotations and the sidesway angles are clockwise.

Substituting the values of displacement above into Equations 4, we establish the member end moments.

$$M_{AB} = -26.45 \text{ kip·ft} \qquad M_{BA} = -21.84 \text{ kip·ft} \qquad \textbf{Ans.}$$

$$M_{BC} = 21.84 \text{ kip·ft} \qquad M_{CB} = 16.78 \text{ kip·ft}$$

$$M_{CD} = -16.76 \text{ kip·ft} \qquad M_{DC} = -18.7 \text{ kip·ft}$$

The final results are summarized in Figure 12.15*b*.

EXAMPLE 12.9

Analyze the frame in Figure 12.16*a* by the slope-deflection method. Given: *EI* is constant for all members.

Solution

Identify the unknown displacements; θ_B, θ_C, and ψ_{AB}. Since the cantilever is a determinate component of the structure, its analysis does not have to be included in the slope-deflection formulation. Instead, we consider the cantilever a device to apply a vertical load of 6 kips and a clockwise moment of 24 kip·ft to joint *C*.

Express member end moments in terms of displacements with Equation 12.16 (all units in kip·feet).

$$M_{AB} = \frac{2EI}{8}(\theta_B - 3\psi_{AB}) - \frac{3(8)^2}{12}$$

$$M_{BA} = \frac{2EI}{8}(2\theta_B - 3\psi_{AB}) + \frac{3(8)^2}{12} \qquad (1)^*$$

**Two additional equations for M_{BC} and M_{CB} on page 468.*

[*continues on next page*]

Example 12.9 continues . . .

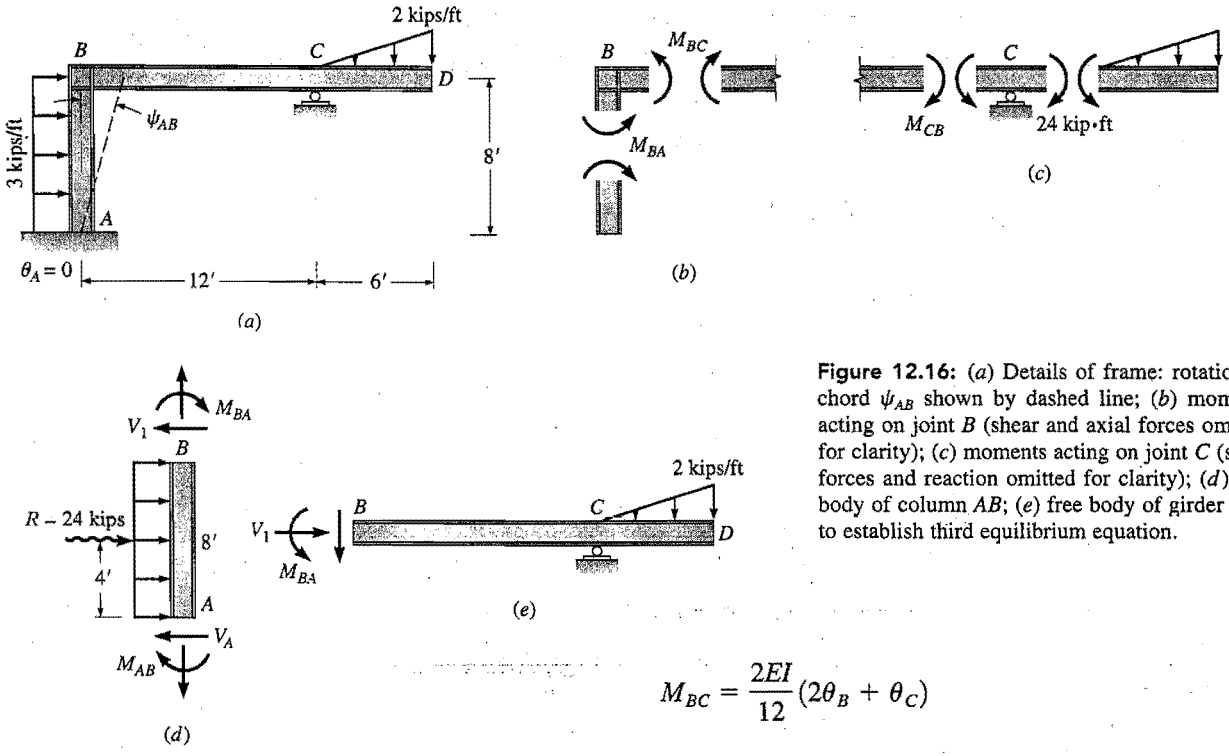


Figure 12.16: (*a*) Details of frame: rotation of chord ψ_{AB} shown by dashed line; (*b*) moments acting on joint *B* (shear and axial forces omitted for clarity); (*c*) moments acting on joint *C* (shear forces and reaction omitted for clarity); (*d*) free body of column *AB*; (*e*) free body of girder used to establish third equilibrium equation.

$$M_{BC} = \frac{2EI}{12}(2\theta_B + \theta_C)$$

$$M_{CB} = \frac{2EI}{12}(2\theta_C + \theta_B)$$

Write the joint equilibrium equations at *B* and *C*.
Joint *B* (see Fig. 12.16*b*):

$$+\circlearrowleft \quad \Sigma M_B = 0: \qquad M_{BA} + M_{BC} = 0 \tag{2}$$

Joint *C* (see Fig. 12.16*c*):

$$+\circlearrowleft \quad \Sigma M_C = 0: \qquad M_{CB} - 24 = 0 \tag{3}$$

Shear equation (see Fig. 12.16*d*):

$$\circlearrowleft^+ \quad \Sigma M_A = 0 \qquad M_{BA} + M_{AB} + 24(4) - V_1(8) = 0$$

$$\text{solving for } V_1 \text{ gives} \qquad V_1 = \frac{M_{BA} + M_{AB} + 96}{8} \tag{4a}$$

Isolate the girder (See Fig. 12.16*e*) and consider equilibrium in the horizontal direction.

$$\rightarrow+ \quad \Sigma F_x = 0: \quad \text{therefore} \quad V_1 = 0 \tag{4b}$$

Substitute Equation 4*a* into Equation 4*b*:

$$M_{BA} + M_{AB} + 96 = 0 \qquad (4)$$

Express equilibrium equations in terms of displacements by substituting Equations 1 into Equations 2, 3, and 4. Collecting terms and simplifying, we find

$$10\theta_B - 2\theta_C - 9\psi_{AB} = -\frac{192}{EI}$$

$$\theta_B - 2\theta_C = \frac{144}{EI}$$

$$3\theta_B - 6\psi_{AB} = -\frac{384}{EI}$$

Solution of the equations above gives

$$\theta_B = \frac{53.33}{EI} \qquad \theta_C = \frac{45.33}{EI} \qquad \psi_{AB} = \frac{90.66}{EI}$$

Establish the values of member end moments by substituting the values of θ_B, θ_C, and ψ_{AB} into Equations 1.

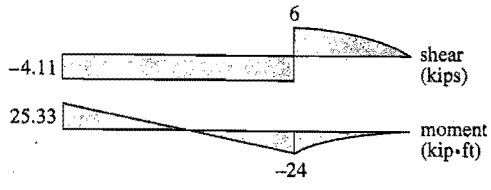
$$M_{AB} = \frac{2EI}{8}\left[\frac{53.33}{EI} - \frac{(3)(90.66)}{EI}\right] - 16 = -70.67 \text{ kip·ft}$$

$$M_{BA} = \frac{2EI}{8}\left[\frac{(2)(53.33)}{EI} - \frac{(3)(90.66)}{EI}\right] + 16 = -25.33 \text{ kip·ft}$$

$$M_{BC} = \frac{2EI}{12}\left[\frac{(2)(53.33)}{EI} + \frac{45.33}{EI}\right] = 25.33 \text{ kip·ft}$$

$$M_{CB} = \frac{2EI}{12}\left[\frac{(2)(45.33)}{EI} + \frac{53.33}{EI}\right] = 24 \text{ kip·ft}$$

After the end moments are established, we compute the shears in all members by applying the equations of equilibrium to free bodies of each member. Final results are shown in Figure 12.16*f*.



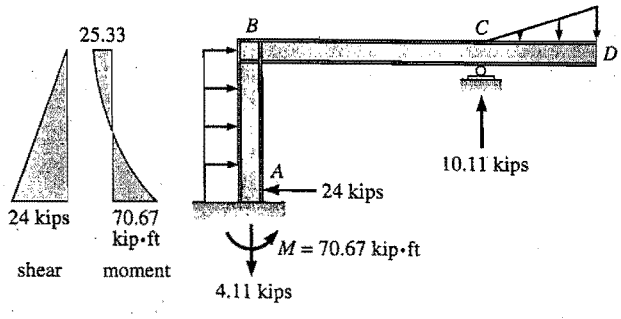


Figure 12.16: (*f*) Reactions and shear and moment curves.

EXAMPLE 12.10

Analyze the frame in Figure 12.17*a* by the slope-deflection method. Determine the reactions, draw the moment curves for the members, and sketch the deflected shape. If $I = 240$ in^4 and $E = 30{,}000$ kips/in^2, determine the horizontal displacement of joint B.

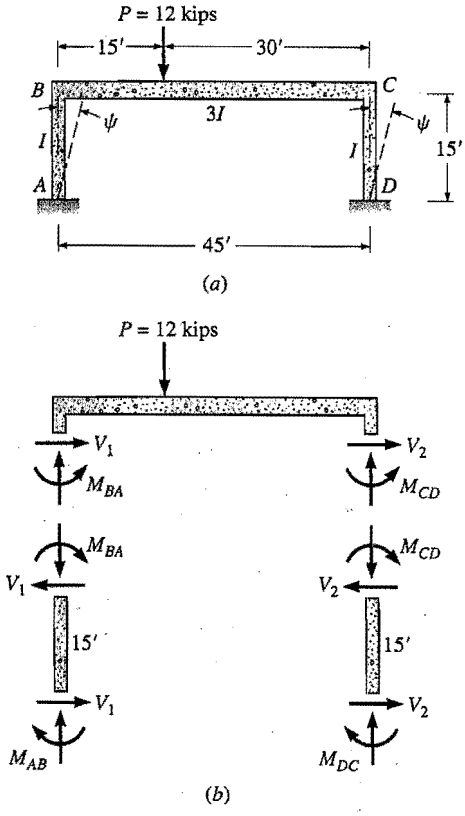


Figure 12.17: (*a*) Unbraced frame positive chord rotations assumed for columns (see the dashed lines), deflected shape shown in (*d*); (*b*) free bodies of columns and girder used to establish the shear equation.

Solution

Unknown displacements are θ_B, θ_C, and ψ. Since supports at A are fixed, θ_A and θ_D equal zero. There is no chord rotation of girder BC.

Express member end moments in terms of displacements with the slope-deflection equation. Use Figure 12.5 to evaluate FEM_{NF}.

$$M_{NF} = \frac{2EI}{L}(2\theta_N + \theta_F - 3\psi_{NF}) + \text{FEM}_{NF} \qquad (12.16)$$

$$\text{FEM}_{BC} = -\frac{Pb^2a}{L^2} = \frac{12(30)^2(15)}{(45)^2} \qquad \text{FEM}_{CD} = \frac{Pa^2b}{L^2} = \frac{12(15)^2(30)}{(45)^2}$$

$$= -80 \text{ kip·ft} \qquad = 40 \text{ kip·ft}$$

To simplify slope-deflection expressions, set $EI/15 = K$.

$$M_{AB} = \frac{2EI}{15}(\theta_B - 3\psi) \qquad = 2K(\theta_B - 3\psi)$$

$$M_{BA} = \frac{2EI}{15}(2\theta_B - 3\psi) \qquad = 2K(2\theta_B - 3\psi)$$

$$M_{BC} = \frac{2EI}{45}(2\theta_B + \theta_C) - 80 = \frac{2}{3}K(2\theta_B + \theta_C) - 80$$

$$M_{CB} = \frac{2EI}{45}(2\theta_C + \theta_B) + 40 = \frac{2}{3}K(2\theta_C + \theta_B) + 40 \qquad (1)$$

$$M_{CD} = \frac{2EI}{15}(2\theta_C - 3\psi) \qquad = 2K(\theta_C - 3\psi)$$

$$M_{DC} = \frac{2EI}{15}(\theta_C - 3\psi) \qquad = 2K(\theta_C - 3\psi)$$

The equilibrium equations are:

Joint B: $$M_{BA} + M_{BC} = 0 \qquad (2)$$

Joint C: $$M_{CB} + M_{CD} = 0 \qquad (3)$$

Shear equation (see the girder in Fig. 12.17*b*):

$$\rightarrow+ \quad \Sigma F_x = 0 \qquad V_1 + V_2 = 0 \qquad (4a)$$

where $$V_1 = \frac{M_{BA} + M_{AB}}{15} \qquad V_2 = \frac{M_{CD} + M_{DC}}{15} \qquad (4b)$$

Substituting V_1 and V_2 given by Equations 4*b* into 4*a* gives

$$M_{BA} + M_{AB} + M_{CD} + M_{DC} = 0 \qquad (4)$$

Alternatively, we can set $Q = 0$ in Equation 12.21 to produce Equation 4.

Express equilibrium equations in terms of displacements by substituting Equations 1 into Equations 2, 3, and 4. Combining terms and simplifying give

$$8K\theta_B + K\theta_C - 9K\psi = 120$$
$$2K\theta_B + 16K\theta_C - 3K\psi = -120$$
$$K\theta_B + K\theta_C - 4K\psi = 0$$

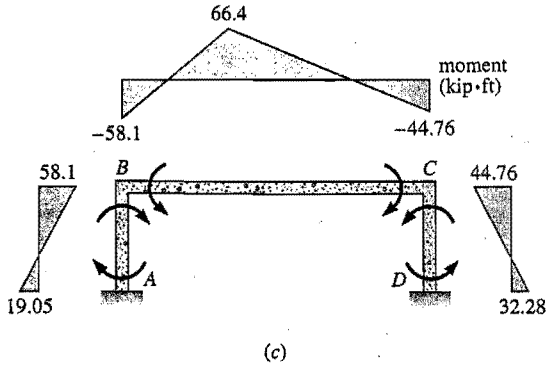
Solving the equations above simultaneously, we compute

$$\theta_B = \frac{410}{21K} \qquad \theta_C = -\frac{130}{21K} \qquad \psi = \frac{10}{3K} \qquad (5)$$

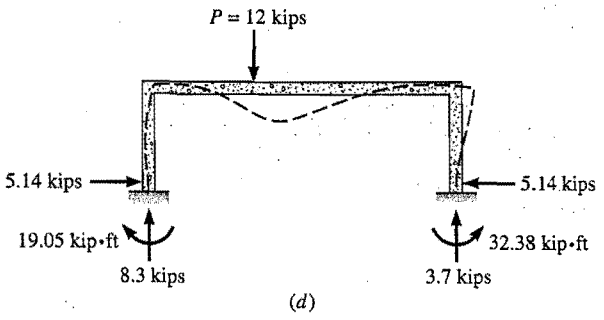
Substituting the values of the θ_B, θ_C, and ψ into Equations 1, we compute the member end moments below.

$$M_{AB} = 19.05 \text{ kip·ft} \qquad M_{BA} = 58.1 \text{ kip·ft}$$
$$M_{CD} = -44.76 \text{ kip·ft} \qquad M_{DC} = -32.38 \text{ kip·ft} \qquad (6)$$
$$M_{BC} = -58.1 \text{ kip·ft} \qquad M_{CB} = 44.76 \text{ kip·ft}$$

Member end moments and moment curves are shown on the sketch in Figure 12.17*c*; the deflected shape is shown in Figure 12.17*d*.



(*c*)



(*d*)

Figure 12.17: (*c*) Member end moments and moment curves (in kip·ft); (*d*) reactions and deflected shape.

[*continues on next page*]

Example 12.10 continues . . .

Compute the horizontal displacement of joint B. Use Equation 1 for M_{AB}. Express all variables in units of inches and kips.

$$M_{AB} = \frac{2EI}{15(12)}(\theta_B - 3\psi) \tag{7}$$

From the values in Equation 5 (p. 485), $\theta_B = 5.86\psi$; substituting into Equation 7, we compute

$$19.05(12) = \frac{2(30{,}000)(240)}{15(12)}(5.86\psi - 3\psi)$$

$$\psi = 0.000999 \text{ rad}$$

$$\psi = \frac{\Delta}{L} \qquad \Delta = \psi L = 0.000999(15 \times 12) = 0.18 \text{ in} \qquad \textbf{Ans.}$$

12.6 Kinematic Indeterminacy

To analyze a structure by the flexibility method, we first established the degree of indeterminacy of the structure. The degree of *statical indeterminacy* determines the number of compatibility equations we must write to evaluate the redundants, which are the unknowns in the compatibility equations.

In the slope-deflection method, displacements—both joint rotations and translations—are the unknowns. As a basic step in this method, we must write equilibrium equations equal in number to the independent joint displacements. The number of independent joint displacements is termed the degree of *kinematic indeterminacy*. To determine the kinematic indeterminacy, we simply count the number of independent joint displacements that are free to occur. For example, if we neglect axial deformations, the beam in Figure 12.18*a* is kinematically indeterminate to the first degree. If we were to analyze this beam by slope-deflection, only the rotation of joint B would be treated as an unknown.

If we also wished to consider axial stiffness in a more general stiffness analysis, the axial displacement at B would be considered an additional unknown, and the structure would be classified as kinematically indeterminate to the second degree. Unless otherwise noted, we will neglect axial deformations in this discussion.

In Figure 12.18*b* the frame would be classified as kinematically indeterminate to the fourth degree because joints A, B, and C are free to rotate

and the girder can translate laterally. Although the number of joint rotations is simple to identify, in certain types of problems the number of independent joint displacements may be more difficult to establish. One method to determine the number of independent joint displacements is to introduce imaginary rollers as joint restraints. The number of rollers required to restrain the joints of the structure from translating equals the number of independent joint displacements. For example, in Figure 12.18*c* the structure would be classified as kinematically indeterminate to the eighth degree, because six joint rotations and two joint displacements are possible. Each imaginary roller (noted by the numbers 1 and 2) introduced at a floor prevents all joints in that floor from displacing laterally. In Figure 12.18*d* the Vierendeel truss would be classified as kinematically indeterminate to the eleventh degree (i.e., eight joint rotations and three independent joint translations). Imaginary rollers (labeled 1, 2, and 3) added at joints *B*, *C*, and *H* prevent all joints from translating.

Summary

- The *slope-deflection* procedure is an early classical method for analyzing indeterminate beams and rigid frames. In this method joint displacements are the unknowns.
- For highly indeterminate structures with a large number of joints, the slope-deflection solution requires that the engineer solve a series of simultaneous equations equal in number to the unknown displacements—a time-consuming operation. While the use of the slope-deflection method to analyze structures is impractical given the availability of computer programs, familiarity with the method provides students with valuable insight into the behavior of structures.
- As an alternate to the slope-deflection method, *moment distribution* was developed in the 1920s to analyze indeterminate beams and frames by distributing unbalanced moments at joints in an artificially restrained structure. While this method eliminates the solution of simultaneous equations, it is still relatively long, especially if a large number of loading conditions must be considered. Nevertheless, moment distribution is a useful tool as an approximate method of analysis both for checking the results of a computer analysis and in making preliminary studies. We will use the slope-deflection equation (in Chap. 13) to develop the moment distribution method.
- A variation of the slope-deflection procedure, the *general stiffness* method, used to prepare general-purpose computer programs, is presented in Chapter 16. This method utilizes stiffness coefficients—forces produced by unit displacements of joints.

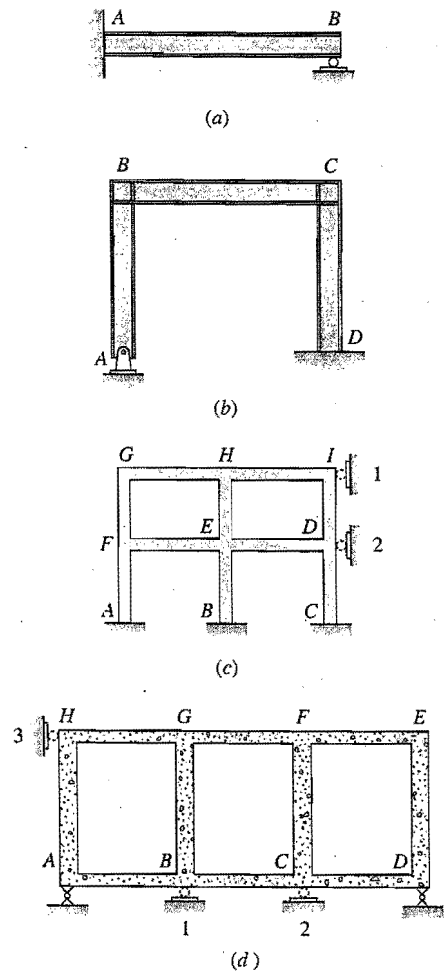
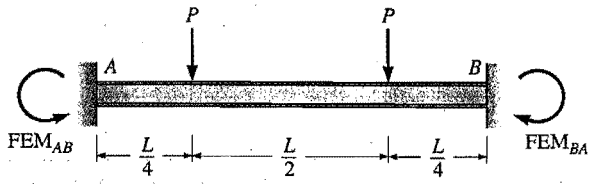


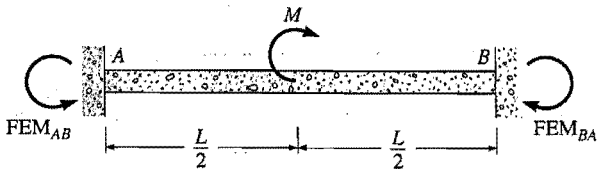
Figure 12.18: Evaluating degree of kinematic indeterminacy: (*a*) indeterminate first degree, neglecting axial deformations; (*b*) indeterminate fourth degree; (*c*) indeterminate eighth degree, imaginary rollers added at points 1 and 2; (*d*) indeterminate eleventh degree, imaginary rollers added at points 1, 2, and 3.

PROBLEMS

P12.1 and **P12.2.** Using Equations 12.12 and 12.13, compute the fixed end moments for the fixed-ended beams. See Figures P12.1 and P12.2.

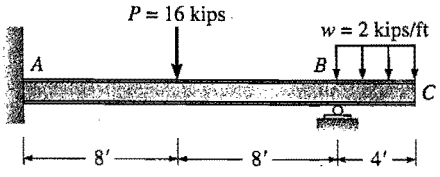


P12.1



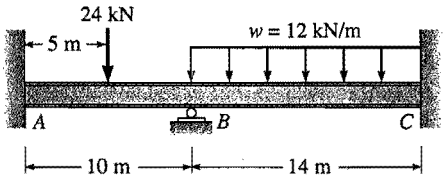
P12.2

P12.3. Analyze by slope-deflection and draw the shear and moment curves for the beam in Figure P12.3. Given: EI = constant.



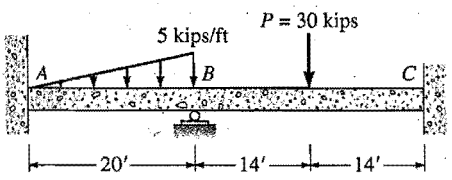
P12.3

P12.4. Analyze the beam in Figure P12.4 by slope-deflection and draw the shear and moment diagrams for the beam. EI is constant.



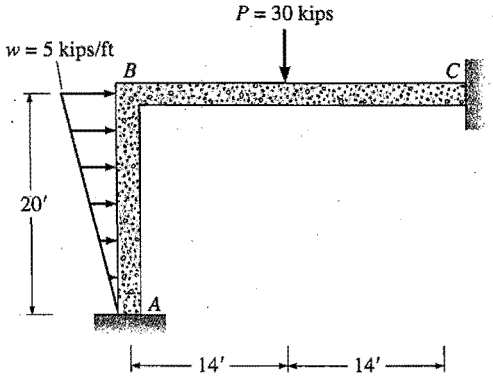
P12.4

P12.5. Analyze by slope-deflection and draw the shear and moment curves for the continuous beam in Figure P12.5. Given: EI is constant.



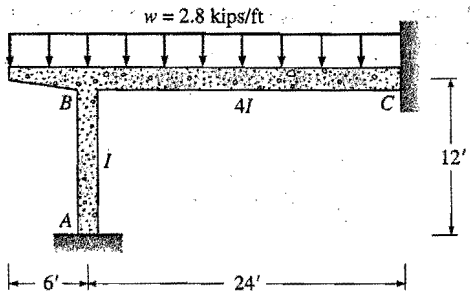
P12.5

P12.6. Draw the shear and moment curves for the frame in Figure P12.6. Given: EI is constant. How does this problem differ from Problem P12.5?



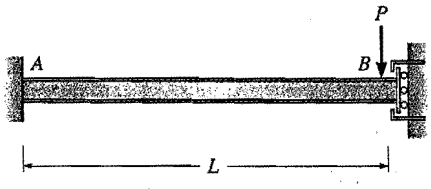
P12.6

P12.7. Compute the reactions at A and C in Figure P12.7. Draw the shear and moment diagram for member BC. Given: $I = 2000$ in^4 and $E = 3{,}000$ kips/in^2.



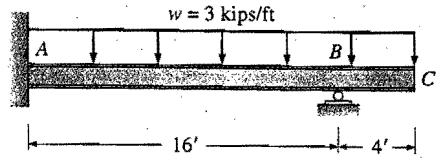
P12.7

P12.8. Use the slope-deflection method to determine the vertical deflection at B and the member end moments at A and B for the beam in Figure P12.8. EI is a constant. The *guide support* at B permits vertical displacement, but allows no rotation or horizontal displacement of the end of the beam.



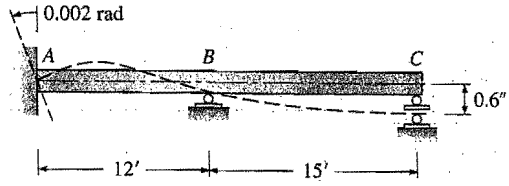
P12.8

P12.9. (a) Under the applied loads support B in Figure P12.9 settles 0.5 in. Determine all reactions. Given: $E = 30{,}000$ kips/in^2, $I = 240$ in^4. (b) Compute the deflection of point C.



P12.9

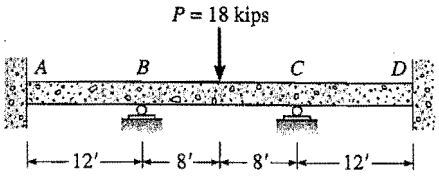
P12.10. In Figure P12.10, support A rotates 0.002 rad and support C settles 0.6 in. Draw the shear and moment curves. Given: $I = 144$ in^4 and $E = 29{,}000$ kips/in^2.



P12.10

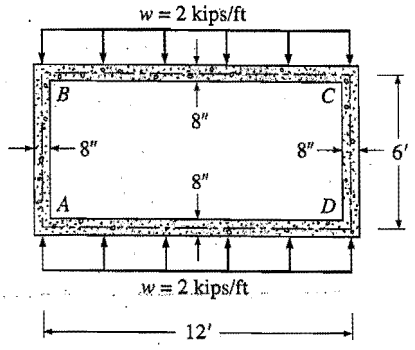
In Problems P12.11 to P12.14, take advantage of symmetry to simplify the analysis by slope deflection.

P12.11. (*a*) Compute all reactions and draw the shear and moment curves for the beam in Figure P12.11. Given: *EI* is constant. (*b*) Compute the deflection under the load.



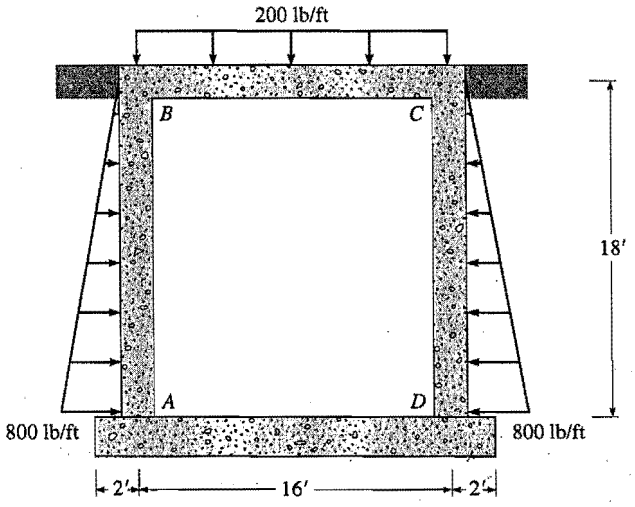
P12.11

P12.12. (*a*) Determine the member end moments for the rectangular ring in Figure P12.12, and draw the shear and moment curves for members *AB* and *AD*. The cross section of the rectangular ring is 12 in × 8 in and $E = 3000$ kips/in^2. (*b*) What is the axial force in member *AD* and in member *AB*?



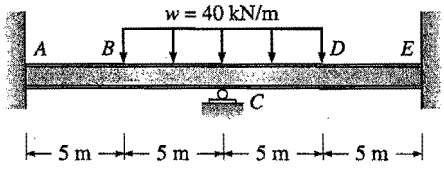
P12.12

P12.13. Figure P12.13 shows the forces exerted by the soil pressure on a typical 1-ft length of a concrete tunnel as well as the design load acting on the top slab. Assume a fixed-end condition at the bottom of the walls at *A* and *D* is produced by the connection to the foundation mat. *EI* is constant.



P12.13

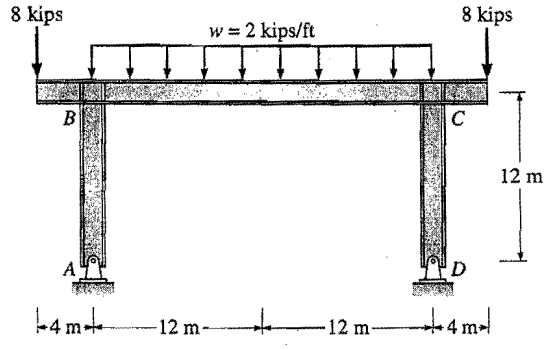
P12.14. Compute the reactions and draw the shear and moment curves for the beam in Figure P12.14. Also $E = 200$ GPa and $I = 120 \times 10^6$ mm^4. Use symmetry to simplify the analysis. Fixed ends at supports *A* and *E*.



P12.14

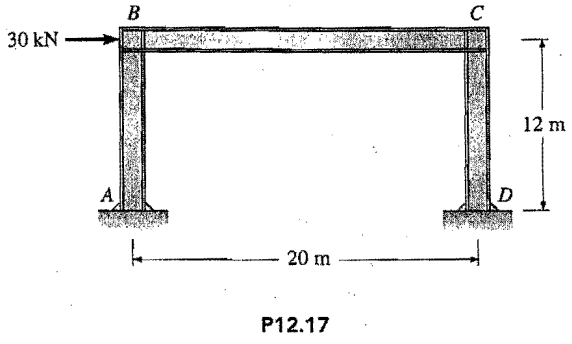
P12.15. Consider the beam in Figure P12.14 without the applied load. Compute the reactions and draw the shear and moment curves for the beam if support C settles 24 mm and support A rotates counterclockwise 0.005 rad.

P12.16. Analyze the frame in Figure P12.16. Given: EI is constant for all members. Use symmetry to simplify the analysis.



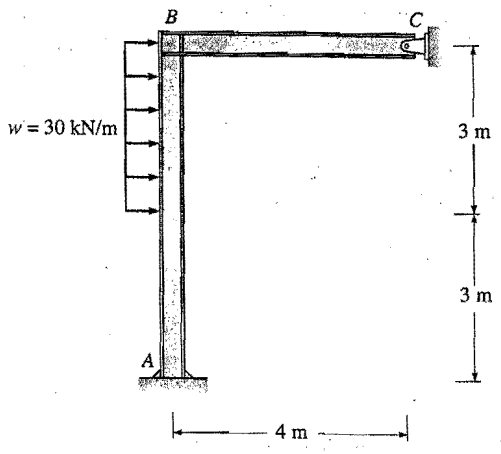
P12.16

P12.17. Analyze the frame in Figure P12.17. Given: EI is constant. Fixed ends at A and D.



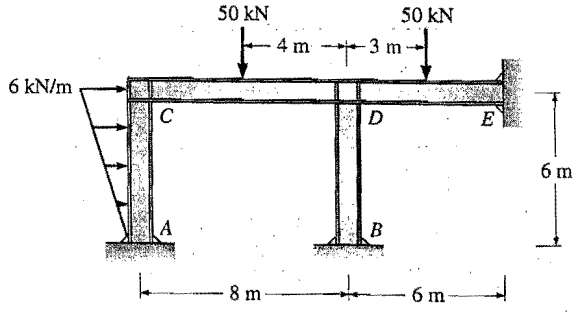
P12.17

P12.18. Analyze the structure in Figure P12.18. In addition to the applied load, support A rotates clockwise by 0.005 rad. Also $E = 200$ GPa and $I = 25 \times 10^6$ mm^4 for all members. Fixed end at A.



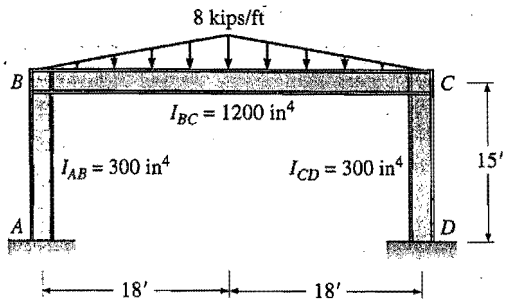
P12.18

P12.19. Analyze the frame in Figure P12.19. Given: EI is constant. Fixed supports at A and B.



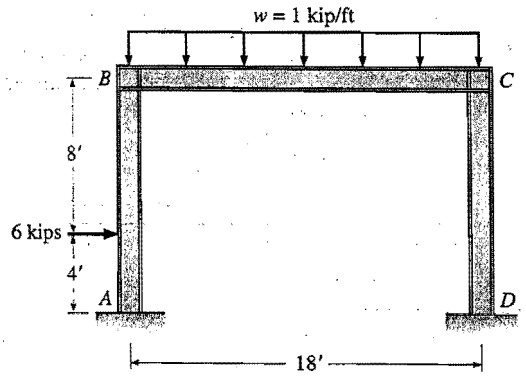
P12.19

P12.20. (*a*) Draw the shear and moment curves for the frame in Figure P12.20. (*b*) Compute the deflection at midspan of girder *BC*. Given: $E = 29{,}000$ kips/in^2.



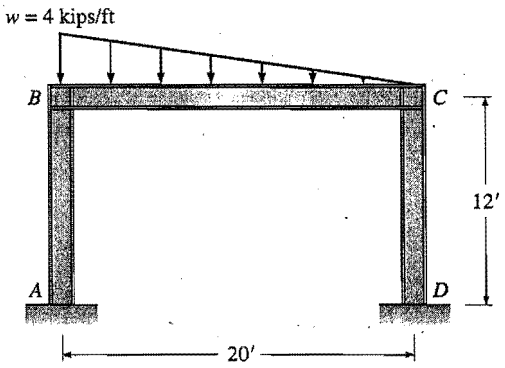
P12.20

P12.21. Analyze the frame in Figure P12.21. Compute all reactions. Also $I_{BC} = 200$ in^4 and $I_{AB} = I_{CD} = 150$ in^4. E is constant.



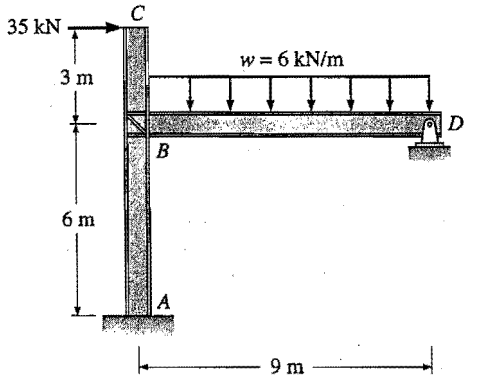
P12.21

P12.22. Analyze the frame in Figure P12.22. Also *EI* is constant. Notice that sidesway is possible because the load is unsymmetric. Compute the horizontal displacement of joint *B*. Given: $E = 29{,}000$ kips/in^2 and $I = 240$ in^4 for all members.



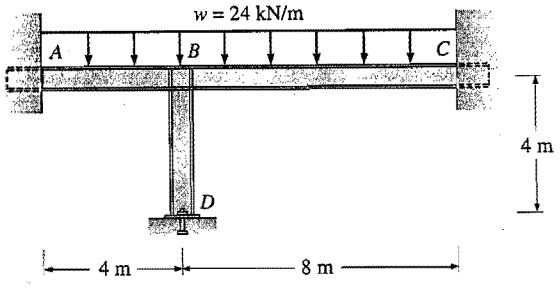
P12.22

P12.23. Compute the reactions and draw the shear and moment diagrams for beam *BC* in Figure P12.23. Also *EI* is constant.



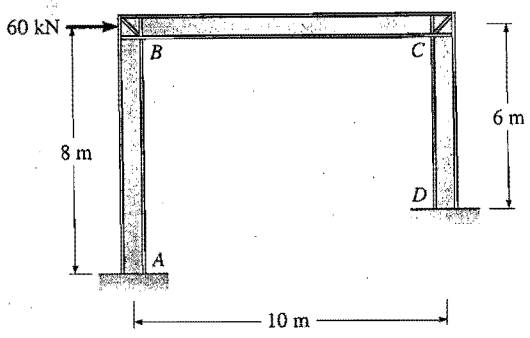
P12.23

P12.24. Determine all reactions in Figure P12.24. Draw the shear and moment diagrams for member *BC*. The ends of the beams at points *A* and *C* are embedded in concrete walls that produce fixed supports. The light baseplate at *D* may be treated as a pin support. Also *EI* is constant.



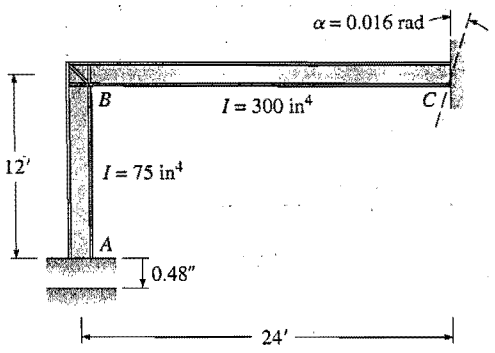
P12.24

P12.25. Determine all reactions at points *A* and *D* in Figure P12.25. *EI* is constant.



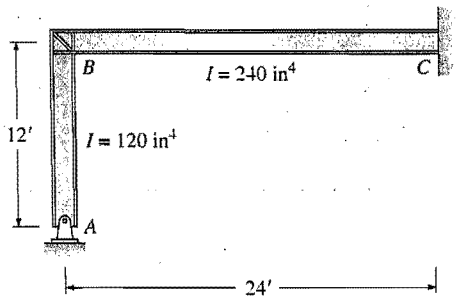
P12.25

P12.26. If support *A* in Figure P12.26 is constructed 0.48 in too low and the support at *C* is accidentally constructed at a slope of 0.016 rad clockwise from a vertical axis through *C*, determine the moment and reactions created when the structure is connected to its supports. Given: $E = 29{,}000$ kips/in^2.



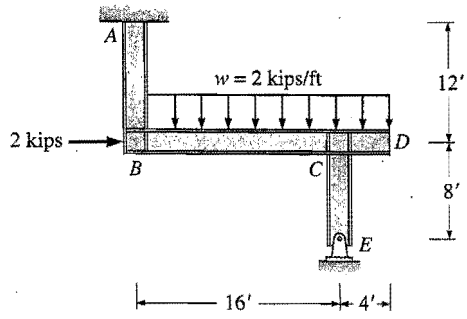
P12.26

P12.27. If member *AB* in Figure P12.27 is fabricated $\frac{3}{4}$ in too long, determine the moments and reactions created in the frame when it is erected. Sketch the deflected shape. $E = 29{,}000$ kips/in^2.



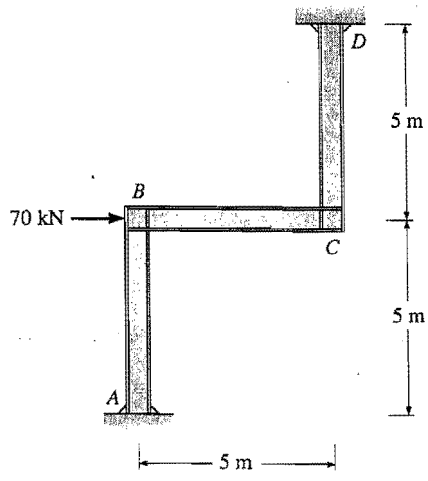
P12.27

P12.28. Set up the equilibrium equations required to analyze the frame in Figure P12.28 by slope deflection. Express the equilibrium equations in terms of the appropriate displacements; *EI* is constant for all members.



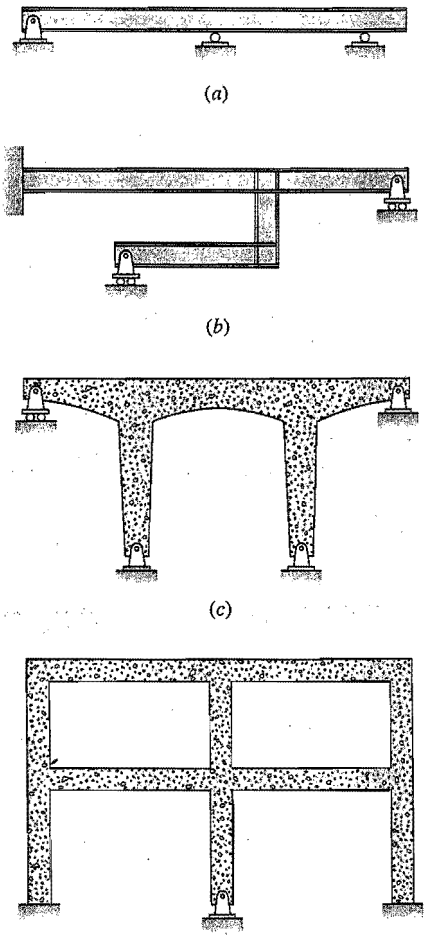
P12.28

P12.29. Analyze the frame in Figure P12.29. Also *EI* is constant. Fixed supports at *A* and *D*.



P12.29

P12.30. Determine the degree of kinematic indeterminacy for each structure in Figure P12.30. Neglect axial deformations.



(*a*)

(*b*)

(*c*)

(*d*)

P12.30

East Bay Drive, a post-tensioned concrete frame bridge, 146 ft long, mainspan 60 ft, edge of concrete girder 7 in thick.