Chapter 12

Analysis of Indeterminate Beams and Frames by the Slope-Deflection Method

12.1 Introduction

The slope-deflection method is a procedure for analyzing indeterminate beams and frames. It is known as a displacement method since equilibrium equations, which are used in the analysis, are expressed in terms of unknown joint displacements.

The slope-deflection method is important because it introduces the student to the stiffness method of analysis. This method is the basis of many general-purpose computer programs for analyzing all types of structures—beams, trusses, shells, and so forth. In addition, moment distribution—a commonly used hand method for analyzing beams and frames rapidly—is also based on the stiffness formulation.

In the slope-deflection method an expression, called the slope-deflection equation, is used to relate the moment at each end of a member both to the end displacements of the member and to the loads applied to the member between its ends. End displacements of a member can include both a rotation and a translation perpendicular to the member’s longitudinal axis.

12.2 Illustration of the Slope-Deflection Method

To introduce the main features of the slope-deflection method, we briefly outline the analysis of a two-span continuous beam. As shown in Figure 12.1a, the structure consists of a single member supported by rollers at points A and B and a pin at C. We imagine that the structure can be divided into beam segments AB and BC and joints A, B, and C by passing planes through the beam an infinitesimal distance before and after each support (see Fig. 12.1b). Since the joints are essentially points in space, the
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Figure 12.1: (a) Continuous beam with applied loads (deflected shape shown by dashed line); (b) free bodies of joints and beams (sign convention: clockwise moment on the end of a member is positive).

length of each member is equal to the distance between joints. In this problem $\theta_A$, $\theta_B$, and $\theta_C$, the rotational displacements of the joints (and also the rotational displacements of the ends of the members), are the unknowns. These displacements are shown to an exaggerated scale by the dashed line in Figure 12.1a. Since the supports do not move vertically, the lateral displacements of the joints are zero; thus there are no unknown joint translations in this example.

To begin the analysis of the beam by the slope-deflection method, we use the **slope-deflection equation** (which we will derive shortly) to express the moments at the ends of each member in terms of the unknown joint displacements and the applied loads. We can represent this step by the following set of equations:

\[
\begin{align*}
M_{AB} &= f(\theta_A, \theta_B, P_1) \\
M_{BA} &= f(\theta_A, \theta_B, P_1) \\
M_{BC} &= f(\theta_B, \theta_C, P_2) \\
M_{CB} &= f(\theta_B, \theta_C, P_2)
\end{align*}
\]  

(12.1)

where the symbol $f(\cdot)$ stands for a function of.
Section 12.3 Derivation of the Slope-Deflection Equation

We next write equilibrium equations that express the condition that the joints are in equilibrium with respect to the applied moments; that is, the sum of the moments applied to each joint by the ends of the beams framing into the joint equals zero. As a sign convention we assume that all unknown moments are positive and act clockwise on the ends of members. Since the moments applied to the ends of members represent the action of the joint on the member, equal and oppositely directed moments must act on the joints (see Fig. 12.1b). The three joint equilibrium equations are

At joint A: \[ M_{AB} = 0 \]
At joint B: \[ M_{BA} + M_{BC} = 0 \] \hspace{1cm} (12.2)
At joint C: \[ M_{CB} = 0 \]

By substituting Equations 12.1 into Equations 12.2, we produce three equations that are functions of the three unknown displacements (as well as the applied loads and properties of the members that are specified). These three equations can then be solved simultaneously for the values of the unknown joint rotations. After the joint rotations are computed, we can evaluate the member end moments by substituting the values of the joint rotations into Equations 12.1. Once the magnitude and direction of the end moments are established, we apply the equations of statics to free bodies of the beams to compute the end shears. As a final step, we compute the support reactions by considering the equilibrium of the joints (i.e., summing forces in the vertical direction).

In Section 12.3 we derive the slope-deflection equation for a typical flexural member of constant cross section using the moment-area method developed in Chapter 9.
percent of the member's buckling load) would not invalidate the derivation. On the other hand, a large compression force would reduce the member's flexural stiffness by creating additional deflection due to the secondary moments produced by the eccentricity of the axial load—the $P$-$\Delta$ effect. As a sign convention, we assume that moments acting at the ends of members in the *clockwise direction* are positive. Clockwise rotations of the ends of members will also be considered positive.

In Figure 12.2c the moment curves produced by both the distributed load $w(x)$ and the end moments $M_{AB}$ and $M_{BA}$ are drawn by parts. The moment curve associated with the distributed load is called the *simple beam moment curve*. In other words, in Figure 12.2c, we are superimposing the moments produced by three loads: (1) the end moment $M_{AB}$, (2) the end moment $M_{BA}$, and (3) the load $w(x)$ applied between ends of the beam. The moment curve for each force has been plotted on the side of the beam that is placed in compression by that particular force.

Figure 12.2d shows the deflected shape of span $AB$ to an *exaggerated scale*. All angles and rotations are shown in the positive sense; that is, all have undergone clockwise rotations from the original horizontal position of the axis. The slope of the chord, which connects the ends of the member at points $A'$ and $B'$ in their deflected position, is denoted by $\psi_{AB}$. To establish if a chord angle is positive or negative, we can draw a horizontal line through either end of the beam. If the horizontal line must be
rotated clockwise through an acute angle to make it coincide with the chord, the slope angle is positive. If a counterclockwise rotation is required, the slope is negative. Notice, in Figure 12.2d, that $\psi_{AB}$ is positive regardless of the end of the beam at which it is evaluated. And $\theta_A$ and $\theta_B$ represent the end rotations of the member. At each end of span $AB$, tangent lines are drawn to the elastic curve; $t_{AB}$ and $t_{BA}$ are the tangential deviations (the vertical distance) from the tangent lines to the elastic curve.

To derive the slope-deflection equation, we will now use the second moment-area theorem to establish the relationship between the member end moments $M_{AB}$ and $M_{BA}$ and the rotational deformations of the elastic curve shown to an exaggerated scale in Figure 12.2d. Since the deformations are small, $\gamma_A$, the angle between the chord and the line tangent to the elastic curve at point $A$, can be expressed as

$$\gamma_A = \frac{t_{BA}}{L}$$  \hspace{1cm} (12.3a)

Similarly, $\gamma_B$, the angle between the chord and the line tangent to the elastic curve at $B$, equals

$$\gamma_B = \frac{t_{AB}}{L}$$  \hspace{1cm} (12.3b)

Since $\gamma_A = \theta_A - \psi_{AB}$ and $\gamma_B = \theta_B - \psi_{AB}$, we can express Equations 12.3a and 12.3b as

$$\theta_A - \psi_{AB} = \frac{t_{BA}}{L}$$  \hspace{1cm} (12.4a)

$$\theta_B - \psi_{AB} = \frac{t_{AB}}{L}$$  \hspace{1cm} (12.4b)

where

$$\psi_{AB} = \frac{\Delta_B - \Delta_A}{L}$$  \hspace{1cm} (12.4c)

To express $t_{AB}$ and $t_{BA}$ in terms of the applied moments, we divide the ordinates of the moment curves in Figure 12.2c by $EI$ to produce $M/EI$ curves and, applying the second moment-area principle, sum the moments of the area under the $M/EI$ curves about the $A$ end of member $AB$ to give $t_{AB}$ and about the $B$ end to give $t_{BA}$.

$$t_{AB} = \frac{M_{BA}}{EI} \frac{L}{2} \frac{2L}{3} - \frac{M_{AB}}{EI} \frac{L}{2} \frac{L}{3} - \frac{(A_M \bar{x})_A}{EI}$$  \hspace{1cm} (12.5)

$$t_{BA} = \frac{M_{AB}}{EI} \frac{L}{2} \frac{2L}{3} - \frac{M_{BA}}{EI} \frac{L}{2} \frac{L}{3} + \frac{(A_M \bar{x})_B}{EI}$$  \hspace{1cm} (12.6)

The first and second terms in Equations 12.5 and 12.6 represent the first moments of the triangular areas associated with the end moments $M_{AB}$ and $M_{BA}$. The last term—$(A_M \bar{x})_A$ in Equation 12.5 and $(A_M \bar{x})_B$ in Equation
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12.6—represents the first moment of the area under the simple beam moment curve about the ends of the beam (the subscript indicates the end of the beam about which moments are taken). As a sign convention, we assume that the contribution of each moment curve to the tangential deviation is positive if it increases the tangential deviation and negative if it decreases the tangential deviation.

To illustrate the computation of \((A_M \vec{x})_A\) for a beam carrying a uniformly distributed load \(w\) (see Fig. 12.3), we draw the simple beam moment curve, a parabolic curve, and evaluate the product of the area under the curve and the distance \(x\) between point \(A\) and the centroid of the area:

\[
(A_M \vec{x})_A = \text{area} \cdot \bar{x} = \frac{2L}{3} \frac{wL^2}{8} \left(\frac{L}{2}\right) = \frac{wL^4}{24} \tag{12.7}
\]

Since the moment curve is symmetric, \((A_M \vec{x})_B\) equals \((A_M \vec{x})_A\).

If we next substitute the values of \(t_{AB}\) and \(t_{BA}\) given by Equations 12.5 and 12.6 into Equations 12.4a and 12.4b, we can write

\[
\theta_A - \psi_{AB} = \frac{1}{L} \left[ \frac{M_{BA}}{El} \frac{2L}{3} \frac{M_{AB}}{El} \frac{L}{2} \frac{L^2}{3} \frac{(A_M \vec{x})_A}{EI} \right] \tag{12.8}
\]

\[
\theta_B - \psi_{AB} = \frac{1}{L} \left[ \frac{M_{AB}}{El} \frac{2L}{3} \frac{M_{BA}}{El} \frac{L}{2} \frac{L^2}{3} \frac{(A_M \vec{x})_B}{EI} \right] \tag{12.9}
\]

To establish the slope-deflection equations, we solve Equations 12.8 and 12.9 simultaneously for \(M_{AB}\) and \(M_{BA}\) to give

\[
M_{AB} = \frac{2EI}{L} (2\theta_A + \theta_B - 3\psi_{AB}) + \frac{2(A_M \vec{x})_A}{L^2} - \frac{4(A_M \vec{x})_B}{L^2} \tag{12.10}
\]

\[
M_{BA} = \frac{2EI}{L} (2\theta_B + \theta_A - 3\psi_{AB}) + \frac{4(A_M \vec{x})_A}{L^2} - \frac{2(A_M \vec{x})_B}{L^2} \tag{12.11}
\]

In Equations 12.10 and 12.11, the last two terms that contain the quantities \((A_M \vec{x})_A\) and \((A_M \vec{x})_B\) are a function of the loads applied between ends of the member only. We can give these terms a physical meaning by using Equations 12.10 and 12.11 to evaluate the moments in a fixed-end beam that has the same dimensions (cross section and span length) and supports the same load as member \(AB\) in Figure 12.2a (see Fig. 12.4). Since the ends of the beam in Figure 12.4 are fixed, the member end moments \(M_{AB}\) and \(M_{BA}\), which are also termed fixed-end moments, may be designated \(FEM_{AB}\) and \(FEM_{BA}\). Because the ends of the beam in Figure 12.4 are fixed against rotation and because no support settlements occur, it follows that

\[
\theta_A = 0 \quad \theta_B = 0 \quad \psi_{AB} = 0
\]
Substituting these values into Equations 12.10 and 12.11 to evaluate the member end moments (or fixed-end moments) in the beam of Figure 12.4, we can write

\[
F_{EM_{AB}} = M_{AB} = \frac{2(A_{Mx})_A}{L^2} - \frac{4(A_{Mx})_B}{L^2} \tag{12.12}
\]

\[
F_{EM_{BA}} = M_{BA} = \frac{4(A_{Mx})_A}{L^2} - \frac{2(A_{Mx})_B}{L^2} \tag{12.13}
\]

Using the results of Equations 12.12 and 12.13, we can write Equations 12.10 and 12.11 more simply by replacing the last two terms by $F_{EM_{AB}}$ and $F_{EM_{BA}}$ to produce

\[
M_{AB} = \frac{2EI}{L} (2\theta_A + \theta_B - 3\psi_{AB}) + F_{EM_{AB}} \tag{12.14}
\]

\[
M_{BA} = \frac{2EI}{L} (2\theta_B + \theta_A - 3\psi_{AB}) + F_{EM_{BA}} \tag{12.15}
\]

Since Equations 12.14 and 12.15 have the same form, we can replace them with a single equation in which we denote the end where the moment is being computed as the near end (N) and the opposite end as the far end (F). With this adjustment we can write the slope-deflection equation as

\[
M_{NF} = \frac{2EI}{L} (2\theta_N + \theta_F - 3\psi_{NF}) + F_{EM_{NF}} \tag{12.16}
\]

In Equation 12.16 the proportions of the member appear in the ratio $I/L$. This ratio, which is called the relative flexural stiffness of member $NF$, is denoted by the symbol $K$.

\[
\text{Relative flexural stiffness } K = \frac{I}{L} \tag{12.17}
\]

Substituting Equation 12.17 into Equation 12.16, we can write the slope-deflection equation as

\[
M_{NF} = 2EK(2\theta_N + \theta_F - 3\psi_{NF}) + F_{EM_{NF}} \tag{12.16a}
\]

The value of the fixed-end moment ($F_{EM_{NF}}$) in Equation 12.16 or 12.16a can be computed for any type of loading by Equations 12.12 and 12.13. The use of these equations to determine the fixed-end moments produced by a single concentrated load at midspan of a fixed-ended beam is illustrated in Example 12.1. See Figure 12.5. Values of fixed-end moments for other types of loading as well as support displacements are also given on the back cover.
EXAMPLE 12.1

Using Equations 12.12 and 12.13, compute the fixed-end moments produced by a concentrated load $P$ at midspan of the fixed-ended beam in Figure 12.6a. We know that $EI$ is constant.

Solution

Equations 12.12 and 12.13 require that we compute, with respect to both ends of the beam in Figure 12.6a, the moment of the area under the simple beam moment curve produced by the applied load. To establish the simple beam moment curve, we imagine the beam $AB$ in Figure 12.6a is removed from the fixed supports and placed on a set of simple supports, as shown in Figure 12.6b. The resulting simple beam moment curve pro-
duced by the concentrated load at midspan is shown in Figure 12.6c. Since the area under the moment curve is symmetric,

\[(A_{Mx})_A = (A_{Mx})_B = \frac{1}{2} \cdot \frac{PL}{4} \left( \frac{L}{2} \right) = \frac{PL^3}{16}\]

Using Equation 12.12 yields

\[FEM_{AB} = \frac{2(A_{Mx})_A}{L^2} - \frac{4(A_{Mx})_B}{L^2} = \frac{2}{L^2} \left( \frac{PL^3}{16} \right) - \frac{4}{L^2} \left( \frac{PL^3}{16} \right) = \frac{PL}{8} \text{ (the minus sign indicates a counterclockwise moment)} \text{ Ans.}\]

Using Equation 12.13 yields

\[FEM_{BA} = \frac{4(A_{Mx})_A}{L^2} - \frac{2(A_{Mx})_B}{L^2} = \frac{4}{L^2} \left( \frac{PL^3}{16} \right) - \frac{2}{L^2} \left( \frac{PL^3}{16} \right) = + \frac{PL}{8} \text{ clockwise Ans. Figure 12.6}\]

12.4 Analysis of Structures by the Slope-Deflection Method

Although the slope-deflection method can be used to analyze any type of indeterminate beam or frame, we will initially limit the method to indeterminate beams whose supports do not settle and to braced frames whose joints are free to rotate but are restrained against the displacement—restraint can be supplied by bracing members (Fig. 3.23g) or by supports. For these types of structures, the chord rotation angle \(\psi_{RF}\) in Equation 12.16 equals zero. Examples of several structures whose joints do not displace laterally but are free to rotate are shown in Figure 12.7a and b. In Figure 12.7a joint A is restrained against displacement by the fixed support and joint C by the pin support. Neglecting second-order changes in the length of members produced by bending and axial deformations, we can assume that joint B is restrained against horizontal displacement by member BC, which is connected to an immovable support at C and against vertical displacement by member AB, which connects to the fixed support at A. The approximate deflected shape of the loaded structures in Figure 12.7 is shown by dashed lines.
Figure 12.7b shows a structure whose configuration and loading are symmetric with respect to the vertical axis passing through the center of member BC. Since a symmetric structure under a symmetric load must deform in a symmetric pattern, no lateral displacement of the top joints can occur in either direction.

Figure 12.7c and d shows examples of frames that contain joints that are free to displace laterally as well as to rotate under the applied loads. Under the lateral load H, joints B and C in Figure 12.7c displace to the right. This displacement produces chord rotations \( \psi = \Delta / h \) in members AB and CD. Since no vertical displacements of joints B and C occur—neglecting second-order bending and axial deformations of the columns—the chord rotation of the girder \( \psi_{BC} \) equals zero. Although the frame in Figure 12.7d supports a vertical load, joints B and C will displace laterally to the right a distance \( \Delta \) because of the bending deformations of members AB and BC. We will consider the analysis of structures that contain one or more members with chord rotations in Section 12.5.

The basic steps of the slope-deflection method, which were discussed in Section 12.2, are summarized briefly below:

**Summary**

1. Identify all unknown joint displacements (rotations) to establish the number of unknowns.
2. Use the slope-deflection equation (Eq. 12.16) to express all member end moments in terms of joint rotations and the applied loads.
3. At each joint, except fixed supports, write the moment equilibrium equation, which states that the sum of the moments (applied by the members framing into the joint) equals zero. An equilibrium equation at a fixed support, which reduces to the identity \( 0 = 0 \), supplies no useful information. The number of equilibrium equations must equal the number of unknown displacements.

As a sign convention, clockwise moments on the ends of the members are assumed to be positive. If a moment at the end of a member is unknown, it must be shown clockwise on the end of a member. The moment applied by a member to a joint is always equal and opposite in direction to the moment acting on the end of the member. If the magnitude and direction of the moment on the end of a member are known, they are shown in the actual direction.

4. Substitute the expressions for moments as a function of displacements (see step 2) into the equilibrium equations in step 3, and solve for the unknown displacements.

Figure 12.7: (a) All joints restrained against displacement; all chord rotations \( \psi \) equal zero; (b) due to symmetry of structure and loading, joints free to rotate but not translate; chord rotations equal zero; (c) and (d) unbraced frames with chord rotations.
5. Substitute the values of displacement in step 4 into the expressions for member end moment in step 2 to establish the value of the member end moments. Once the member end moments are known, the balance of the analysis—drawing shear and moment curves or computing reactions, for example—is completed by statics.

Examples 12.2 and 12.3 illustrate the procedure outlined above.

Using the slope-deflection method, determine the member end moments in the indeterminate beam shown in Figure 12.8a. The beam, which behaves elastically, carries a concentrated load at midspan. After the end moments are determined, draw the shear and moment curves. If \( I = 240 \text{ in}^4 \) and \( E = 30,000 \text{ kips/in}^2 \), compute the magnitude of the slope at joint \( B \).

**Example 12.2**

Using the slope-deflection method, determine the member end moments in the indeterminate beam shown in Figure 12.8a. The beam, which behaves elastically, carries a concentrated load at midspan. After the end moments are determined, draw the shear and moment curves. If \( I = 240 \text{ in}^4 \) and \( E = 30,000 \text{ kips/in}^2 \), compute the magnitude of the slope at joint \( B \).

**Figure 12.8:** (a) Beam with one unknown displacement \( \theta_B \); (b) free body of beam \( AB \); unknown member end moments \( M_{AB} \) and \( M_{BA} \) shown clockwise; (c) free body of joint \( B \); (d) free body used to compute end shears; (e) shear and moment curves.

[continues on next page]


Example 12.2 continues...

**Solution**

Since joint A is fixed against rotation, \( \theta_A = 0 \); therefore, the only unknown displacement is \( \theta_B \), the rotation of joint B (\( \psi_{AB} \) is, of course, zero since no support settlements occur). Using the slope-deflection equation

\[
M_{NF} = \frac{2EI}{L} \left(2\theta_N + \theta_F - 3\psi_{NF}\right) + \text{REM}_{NF} \tag{12.16}
\]

and the values in Figure 12.5a for the fixed-end moments produced by a concentrated load at midspan, we can express the member end moments shown in Figure 12.8b as

\[
M_{AB} = \frac{2EI}{L} \left(\theta_B\right) - \frac{PL}{8} \tag{1}
\]

\[
M_{BA} = \frac{2EI}{L} \left(2\theta_B\right) + \frac{PL}{8} \tag{2}
\]

To determine \( \theta_B \), we next write the equation of moment equilibrium at joint B (see Fig. 12.8c):

\[
\sum M_B = 0
\]

\[
M_{BA} = 0 \tag{3}
\]

Substituting the value of \( M_{BA} \) given by Equation 2 into Equation 3 and solving for \( \theta_B \) give

\[
\frac{4EI}{L} \theta_B + \frac{PL}{8} = 0
\]

\[
\theta_B = -\frac{PL^2}{32EI} \tag{4}
\]

where the minus sign indicates both that the B end of member AB and joint B rotate in the counterclockwise direction. To determine the member end moments, the value of \( \theta_B \) given by Equation 4 is substituted into Equations 1 and 2 to give

\[
M_{AB} = \frac{2EI}{L} \left(-\frac{PL^2}{32EI}\right) - \frac{PL}{8} = -\frac{3PL}{16} = -54 \text{ kip-ft} \quad \text{Ans.}
\]

\[
M_{BA} = \frac{4EI}{L} \left(-\frac{PL^2}{32EI}\right) + \frac{PL}{8} = 0
\]

Although we know that \( M_{BA} \) is zero since the support at B is a pin, the computation of \( M_{BA} \) serves as a check.

To complete the analysis, we apply the equations of statics to a free body of member AB (see Fig. 12.8d).
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\[ \sum M_A = 0 \]
\[ 0 = (16 \text{kips})(9 \text{ft}) - V_{BA}(18 \text{ft}) - 54 \text{kip-ft} \]
\[ V_{BA} = 5 \text{kips} \]

\[ \sum F_Y = 0 \]
\[ 0 = V_{BA} + V_{AB} - 16 \]
\[ V_{AB} = 11 \text{kips} \]

To evaluate \( \theta_B \), we express all variables in Equation 4 in units of inches and kips.

\[ \theta_B = \frac{-PL^2}{32EI} = \frac{-16(18 \times 12)^2}{32(30,000)240} = -0.0032 \text{ rad} \]

Expressing \( \theta_B \) in degrees, we obtain

\[ \frac{2\pi \text{ rad}}{360^\circ} = -0.0032 \]
\[ \theta_B = -0.183^\circ \]

\text{Ans.}

Note that the slope \( \theta_B \) is extremely small and not discernible to the naked eye.

\text{NOTE.} When you analyze a structure by the slope-deflection method, you must follow a rigid format in formulating the equilibrium equations. There is no need to guess the direction of unknown member end moments since the solution of the equilibrium equations will automatically produce the correct direction for displacements and moments. For example, in Figure 12.8b we show the moments \( M_{AB} \) and \( M_{BA} \) clockwise on the ends of member \( AB \) even though intuitively we may recognize from a sketch of the deflected shape in Figure 12.8a that moment \( M_{AB} \) must act in the counterclockwise direction because the beam is bent concave downward at the left end by the load. When the solution indicates \( M_{AB} = -54 \text{ kip-ft} \), we know from the negative sign that \( M_{AB} \) actually acts on the end of the member in the counterclockwise direction.

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Using the slope-deflection method, determine the member end moments in the braced frame shown in Figure 12.9a. Also compute the reactions at support \( D \), and draw the shear and moment curves for members \( AB \) and \( BD \).
Example 12.3 continues . . .

Figure 12.9: (a) Frame details; (b) joint D; (c) joint B (shears and axial forces omitted for clarity); (d) free bodies of members and joints used to compute shears and reactions (moments acting on joint B omitted for clarity).
Solution
Since $\theta_A$ equals zero because of the fixed support at $A$, $\theta_B$ and $\theta_D$ are the only unknown joint displacements we must consider. Although the moment applied to joint $B$ by the cantilever $BC$ must be included in the joint equilibrium equation, there is no need to include the cantilever in the slope-deflection analysis of the indeterminate portions of the frame because the cantilever is determinate; that is, the shear and the moment at any section of member $BC$ can be determined by the equations of statics. In the slope-deflection solution, we can treat the cantilever as a device that applies a vertical force of 6 kips and a clockwise moment of 24 kip-ft to joint $B$.

Using the slope-deflection equation

$$M_{NF} = \frac{2EI}{L} (2\theta_N + \theta_F - 3\psi_{NF}) + FEM_{NF} \quad (12.16)$$

where all variables are expressed in units of kip-inches and the fixed-end moments produced by the uniform load on member $AB$ (see Fig. 12.5d) equal

$$FEM_{AB} = -\frac{wL^2}{12}$$

$$FEM_{BA} = +\frac{wL^2}{12}$$

we can express the member end moments as

$$M_{AB} = \frac{2E(120)}{18(12)} (\theta_B) - \frac{2(18)^2(12)}{12} = 1.11E\theta_B - 648 \quad (1)$$

$$M_{BA} = \frac{2E(120)}{18(12)} (2\theta_B) + \frac{2(18)^2(12)}{12} = 2.22E\theta_B + 648 \quad (2)$$

$$M_{BD} = \frac{2E(60)}{9(12)} (2\theta_D + \theta_B) = 2.22E\theta_D + 1.11E\theta_B \quad (3)$$

$$M_{DB} = \frac{2E(60)}{9(12)} (2\theta_D + \theta_B) = 2.22E\theta_D + 1.11E\theta_B \quad (4)$$

To solve for the unknown joint displacements $\theta_B$ and $\theta_D$, we write equilibrium equations at joints $D$ and $B$.

At joint $D$ (see Fig. 12.9b): $$\Sigma M_D = 0 \quad M_{DB} = 0 \quad (5)$$

At joint $B$ (see Fig. 12.9c): $$\Sigma M_B = 0 \quad M_{BA} + M_{BD} - 24(12) = 0 \quad (6)$$

[continues on next page]
Example 12.3 continues . . .

Since the magnitude and direction of the moment $M_{BC}$ at the $B$ end of the cantilever can be evaluated by statics (summing moments about point $B$), it is applied in the correct sense (counterclockwise) on the end of member $BC$, as shown in Figure 12.9c. On the other hand, since the magnitude and direction of the end moments $M_{BA}$ and $M_{BD}$ are unknown, they are assumed to act in the positive sense—clockwise on the ends of the members and counterclockwise on the joint.

Using Equations 2 to 4 to express the moments in Equations 5 and 6 in terms of displacements, we can write the equilibrium equations as

At joint $D$:

\[ 2.22E\theta_D + 1.11E\theta_B = 0 \quad (7) \]

At joint $B$: \[ (2.22E\theta_B + 648) + (2.22E\theta_B + 1.11E\theta_D) - 288 = 0 \quad (8) \]

Solving Equations 7 and 8 simultaneously gives

\[ \theta_D = \frac{46.33}{E} \]
\[ \theta_B = \frac{92.66}{E} \]

To establish the values of the member end moments, the values of $\theta_B$ and $\theta_D$ above are substituted into Equations 1, 2, and 3, giving

\[ M_{AB} = 1.11E\left(-\frac{92.66}{E}\right) - 648 \]
\[ = -750.85 \text{ kip·in} = -62.57 \text{ kip·ft} \quad \text{Ans.} \]

\[ M_{BA} = 2.22E\left(-\frac{92.66}{E}\right) + 648 \]
\[ = 442.29 \text{ kip·in} = +36.86 \text{ kip·ft} \quad \text{Ans.} \]

\[ M_{BD} = 2.22E\left(-\frac{92.66}{E}\right) + 1.11E\left(\frac{46.33}{E}\right) \]
\[ = -154.28 \text{ kip·in} = -12.86 \text{ kip·ft} \quad \text{Ans.} \]

Now that the member end moments are known, we complete the analysis by using the equations of statics to determine the shears at the ends of all members. Figure 12.9d shows free-body diagrams of both members and joints: Except for the cantilever, all members carry axial forces as well as shear and moment. After the shears are computed, axial forces and reactions can be evaluated by considering the equilibrium of the joints. For example, vertical equilibrium of the forces applied to joint $B$ requires that the vertical force $F$ in column $BD$ equal the sum of the shears applied to joint $B$ by the $B$ ends of members $AB$ and $BC$. 
Use of Symmetry to Simplify the Analysis of a Symmetric Structure with a Symmetric Load

Determine the reactions and draw the shear and moment curves for the columns and girder of the rigid frame shown in Figure 12.10a. Given: \( I_{AB} = I_{CD} = 120 \text{ in}^4, I_{BC} = 360 \text{ in}^4, \) and \( E \) is constant for all members.

Solution

Although joints \( B \) and \( C \) rotate, they do not displace laterally because both the structure and its load are symmetric with respect to a vertical axis of symmetry passing through the center of the girder. Moreover, \( \theta_B \) and \( \theta_C \) are equal in magnitude; however, \( \theta_B \), a clockwise rotation, is positive,

\[
\begin{align*}
M_{BC} &= M_{BA} \\
M_{CD} &= M_{DA}
\end{align*}
\]

\[ \theta_B = \theta, \quad \theta_C = -\theta \]

\[ w = 2 \text{ kips/ft} \]

\[ V = 30 \text{ kips} \]

\[ w = 2 \text{ kips/ft} \]

\[ V = 30 \text{ kips} \]

\[ \begin{align*}
7.81 \text{ kips} &\quad 83.33 \text{ kip-ft} \\
\text{shear} &\quad \text{moment}
\end{align*} \]

\[ 30 \text{ kips} \]

\[ 41.67 \text{ kip-ft} \]

\[ 30 \text{ kips} \]

\[ 7.81 \text{ kips} \]

\[ 83.33 \text{ kip-ft} \]

\[ \begin{align*}
A_B &= 7.81 \text{ kips} \\
A_C &= 30 \text{ kips}
\end{align*} \]

\[ \begin{align*}
A_B &= 7.81 \text{ kips} \\
A_C &= 30 \text{ kips}
\end{align*} \]

\[ \begin{align*}
7.81 \text{ kips} &\quad 83.33 \text{ kip-ft} \\
\text{shear} &\quad \text{moment}
\end{align*} \]

\[ \begin{align*}
7.81 \text{ kips} &\quad 83.33 \text{ kip-ft} \\
\text{shear} &\quad \text{moment}
\end{align*} \]
Example 12.4 continues...

and \( \theta_c \), a counterclockwise rotation, is negative. Since the problem contains only one unknown joint rotation, we can determine its magnitude by writing the equilibrium equation for either joint \( B \) or joint \( C \). We will arbitrarily choose joint \( B \).

Expressing member end moments with Equation 12.16, reading the value of fixed-end moment for member \( BC \) from Figure 12.5d, expressing units in kips-inch, and substituting \( \theta_B = \theta \) and \( \theta_C = -\theta \), we can write

\[
M_{AB} = \frac{2E(120)}{16(12)} \theta_B = 1.25E\theta_B \quad (1)
\]

\[
M_{BA} = \frac{2E(120)}{16(12)} (2\theta_B) = 2.50E\theta_B \quad (2)
\]

\[
M_{BC} = \frac{2E(360)}{30(12)} (2\theta_B + \theta_C) - \frac{wL^2}{12}
= 2E [2\theta + (-\theta)] - \frac{2(30)^2(12)}{12} = 2E\theta - 1800 \quad (3)
\]

Writing the equilibrium equation at joint \( B \) (see Fig. 12.10b) yields

\[
M_{BA} + M_{BC} = 0 \quad (4)
\]

Substituting Equations 2 and 3 into Equation 4 and solving for \( \theta \) produce

\[
2.5E\theta + 2.0E\theta - 1800 = 0
\]

\[
\theta = \frac{400}{E} \quad (5)
\]

Substituting the value of \( \theta \) given by Equation 5 into Equations 1, 2, and 3 gives

\[
M_{AB} = 1.25E \left( \frac{400}{E} \right)
= 500 \text{ kip-in} = 41.67 \text{ kip-ft} \quad \text{Ans.}
\]

\[
M_{BA} = 2.5E \left( \frac{400}{E} \right)
= 1000 \text{ kip-in} = 83.33 \text{ kip-ft} \quad \text{Ans.}
\]

\[
M_{BC} = 2E \left( \frac{400}{E} \right) - 1800
= -1000 \text{ kip-in} = -83.33 \text{ kip-ft counterclockwise} \quad \text{Ans.}
\]

The final results of the analysis are shown in Figure 12.10c.
Using symmetry to simplify the slope-deflection analysis of the frame in Figure 12.11a, determine the reactions at supports A and D.

Solution
Examination of the frame shows that all joint rotations are zero. Both $\theta_A$ and $\theta_C$ are zero because of the fixed supports at A and C. Since column BD lies on the vertical axis of symmetry, we can infer that it must remain straight since the deflected shape of the structure with respect to the axis of symmetry must be symmetric. If the column were to bend in either direction, the requirement that the pattern of deformations be symmetric...
would be violated. Since the column remains straight, neither the top nor bottom joints at B and D rotate; therefore, both \( \theta_B \) and \( \theta_D \) equal zero. Because no support settlements occur, chord rotations for all members are zero. Since all joint and chord rotations are zero, we can see from the slope-deflection equation (Eq. 12.16) that the member end moments at each end of beams \( AB \) and \( BC \) are equal to the fixed-end moments \( PL/8 \) given by Figure 12.5a:

\[
FEM = \pm \frac{PL}{8} = \frac{16(20)}{8} = \pm 40 \text{ kip-ft}
\]

Free bodies of beam \( AB \), joint \( B \), and column \( BD \) are shown in Figure 12.11.

NOTE. The analysis of the frame in Figure 12.11 shows that column \( BD \) carries only axial load because the moments applied by the beams to each side of the joint are the same. A similar condition often exists at the interior columns of multistory buildings whose structure consists of either a continuous reinforced concrete or a welded-steel rigid-jointed frame. Although a rigid joint has the capacity to transfer moments from the beams to the column, it is the difference between the moments applied by the girders on either side of a joint that determines the moment to be transferred. When the span lengths of the beams and the loads they support are approximately the same (a condition that exists in most buildings), the difference in moment is small. As a result, in the preliminary design stage most columns can be sized accurately by considering only the magnitude of the axial load produced by the gravity load from the tributary area supported by the column.

**Example 12.6**

Determine the reactions and draw the shear and moment curves for the beam in Figure 12.12. The support at \( A \) has been accidentally constructed with a slope that makes an angle of 0.009 rad with the vertical y-axis through support \( A \), and \( B \) has been constructed 1.2 in below its intended position. Given: \( EI \) is constant, \( I = 360 \text{ in}^4 \), and \( E = 29,000 \text{ kips/in}^2 \).

**Solution**

The slope at \( A \) and the chord rotation \( \psi_{AB} \) can be determined from the information supplied about the support displacements. Since the end of the beam is rigidly connected to the fixed support at \( A \), it rotates counterclockwise with the support; and \( \theta_A = -0.009 \text{ rad} \). The settlement of support \( B \) relative to support \( A \) produces a clockwise chord rotation

\[
\psi_{AB} = \frac{\Delta}{L} = \frac{1.2}{20(12)} = 0.005 \text{ radians}
\]
Angle $\theta_B$ is the only unknown displacement, and the fixed-end moments are zero because no loads act on beam. Expressing member end moments with the slope-deflection equation (Eq. 12.16), we have

$$M_{AB} = \frac{2EI_{AB}}{L_{AB}} (2\theta_A + \theta_B - 3\psi_{AB}) + FEM_{AB}$$

$$M_{AB} = \frac{2E(360)}{20(12)} [2(-0.009) + \theta_B - 3(0.005)]$$  (1)

$$M_{BA} = \frac{2E(360)}{20(12)} [2\theta_B + (-0.009) - 3(0.005)]$$  (2)

Writing the equilibrium equation at joint $B$ yields

$$\sum M_B = 0$$

$$M_{BA} = 0$$  (3)

Substituting Equation 2 into Equation 3 and solving for $\theta_B$ yield

$$3E(2\theta_B - 0.009 - 0.015) = 0$$

$$\theta_B = 0.012 \text{ radians}$$

To evaluate $M_{AB}$, substitute $\theta_B$ into Equation 1:

$$M_{AB} = 3(29,000) [2(-0.009) + 0.012 - 3(0.005)]$$

$$= -1827 \text{ kip·in} = -152.25 \text{ kip·ft}$$

Complete the analysis by using the equations of statics to compute the reaction at $B$ and the shear at $A$ (see Fig. 12.12b).

$$\sum M_A = 0$$

$$0 = R_B(20) - 152.25$$

$$R_B = 7.61 \text{ kips} \hspace{1cm} \text{Ans.}$$

$$\sum F_y = 0$$

$$V_A = 7.61 \text{ kips}$$

Although the supports are constructed in their correct position, girder $AB$ of the frame shown in Figure 12.13 is fabricated 1.2 in too long. Determine the reactions created when the frame is connected into the supports. Given: $EI$ is a constant for all members, $I = 240 \text{ in}^4$, and $E = 29,000 \text{ kips/in}^2$. [continues on next page]
Example 12.7 continues...

Figure 12.13: (a) Girder $AB$ fabricated 1.2 in too long; (b) free-body diagrams of beam $AB$, joint $B$, and column $BC$ used to compute internal forces and reactions.

Solution

The deflected shape of the frame is shown by the dashed line in Figure 12.13a. Although internal forces (axial, shear, and moment) are created when the frame is forced into the supports, the deformations produced by these forces are neglected since they are small compared to the 1.2-in fabrication error; therefore, the chord rotation $\psi_{BC}$ of column $BC$ equals

$$\psi_{BC} = \frac{\Delta}{L} = \frac{1.2}{9(12)} = \frac{1}{90} \text{ rad}$$

Since the ends of girder $AB$ are at the same level, $\psi_{AB} = 0$. The unknown displacements are $\theta_A$ and $\theta_C$. 
Using the slope-deflection equation (Eq. 12.16), we express member end moments in terms of the unknown displacements. Because no loads are applied to the members, all fixed-end moments equal zero.

\[
M_{AB} = \frac{2E(240)}{18(12)} (\theta_B) = 2.222E\theta_B \\
M_{BA} = \frac{2E(240)}{18(12)} (2\theta_B) = 4.444E\theta_B \\
M_{BC} = \frac{2E(240)}{9(12)} \left[ 2\theta_B + \theta_C - 3 \left( \frac{1}{90} \right) \right] \\
= 8.889E\theta_B + 4.444E\theta_C - 0.1481E \\
M_{CB} = \frac{2E(240)}{9(12)} \left[ 2\theta_C + \theta_B - 3 \left( \frac{1}{90} \right) \right] \\
= 8.889E\theta_C + 4.444E\theta_B - 0.1481E
\]

Writing equilibrium equations gives

Joint C: \[ M_{CB} = 0 \] (5)  
Joint B: \[ M_{BA} + M_{BC} = 0 \] (6)

Substituting Equations 2 to 4 into Equations 5 and 6 solving for \( \theta_B \) and \( \theta_C \) yield

\[ 8.889E\theta_C + 4.444E\theta_B - 0.1481E = 0 \]
\[ 4.444E\theta_B + 8.889E\theta_B + 4.444E\theta_C - 0.1481E = 0 \]
\[ \theta_B = 0.00666 \text{ rad} \] (7)  
\[ \theta_C = 0.01332 \text{ rad} \] (8)

Substituting \( \theta_C \) and \( \theta_B \) into Equations 1 to 3 produces

\[ M_{AB} = 35.76 \text{ kip-ft} \quad M_{BA} = 71.58 \text{ kip-ft} \quad \text{Ans.} \]
\[ M_{BC} = -71.58 \text{ kip-ft} \quad M_{CB} = 0 \]

The free-body diagrams used to compute internal forces and reactions are shown in Figure 12.13b, which also shows moment diagrams.
whose joints are also free to sidesway, that is, to displace laterally. For example, in Figure 12.14a the horizontal load results in girder BC displacing laterally a distance $\Delta$. Recognizing that the axial deformation of the girder is insignificant, we assume that the horizontal displacement of the top of both columns equals $\Delta$. This displacement creates a clockwise chord rotation $\psi$ in both legs of the frame equal to

$$\psi = \frac{\Delta}{h}$$

where $h$ is the length of column.

Since three independent displacements develop in the frame [i.e., the rotation of joints B and C ($\theta_B$ and $\theta_C$) and the chord rotation $\psi$], we require three equilibrium equations for their solution. Two equilibrium equations are supplied by considering the equilibrium of the moments acting on joints B and C. Since we have written equations of this type in the solution of previous problems, we will only discuss the second type of equilibrium equation—the shear equation. The shear equation is established by summing in the horizontal direction the forces acting on a free body of the girder. For example, for the girder in Figure 12.14b we can write

$$-H \sum F_x = 0$$

$$V_1 + V_2 + Q = 0 \quad (12.18)$$

In Equation 12.18, $V_1$, the shear in column $AB$, and $V_2$, the shear in column $CD$, are evaluated by summing moments about the bottom of each column of the forces acting on a free body of the column. As we established previously, the unknown moments on the ends of the column must always be shown in the positive sense, that is, acting clockwise on the end of the member. Summing moments about point A of column $AB$, we compute $V_1$:

$$C^+ \sum M_A = 0$$

$$M_{AB} + M_{BA} - V_1 h = 0$$

$$V_1 = \frac{M_{AB} + M_{BA}}{h} \quad (12.19)$$

Similarly, the shear in column $CD$ is evaluated by summing moments about point D.

$$C^+ \sum M_D = 0$$

$$M_{CD} + M_{DC} - V_2 h = 0$$

$$V_2 = \frac{M_{CD} + M_{DC}}{h} \quad (12.20)$$
Substituting the values of \( V_1 \) and \( V_2 \) from Equations 12.19 and 12.20 into Equation 12.18, we can write the third equilibrium equation as

\[
\frac{M_{AB} + M_{BA}}{h} + \frac{M_{CD} + M_{DC}}{h} + Q = 0 \quad (12.21)
\]

Examples 12.8 and 12.9 illustrate the use of the slope-deflection method to analyze frames that carry lateral loads and are free to sidesway. Frames that carry only vertical load will also undergo small amounts of sidesway unless both the structure and the loading pattern are symmetric. Example 12.10 illustrates this case.

**Example 12.8**

Analyze the frame in Figure 12.15a by the slope-deflection method. \( E \) is constant for all members.

\[
\begin{align*}
I_{AB} &= 240 \text{ in}^4 \\
I_{BC} &= 600 \text{ in}^4 \\
I_{CD} &= 360 \text{ in}^4
\end{align*}
\]

**Solution**

Identify the unknown displacements \( \theta_B, \theta_C, \) and \( \Delta \). Express the chord rotations \( \psi_{AB} \) and \( \psi_{CD} \) in terms of \( \Delta \):

\[
\psi_{AB} = \frac{\Delta}{12} \quad \text{and} \quad \psi_{CD} = \frac{\Delta}{18} \quad \text{so} \quad \psi_{AB} = 1.5\psi_{CD} \quad (1)
\]

Figure 12.15: (a) Details of frame; (b) reactions and moment diagrams.
Example 12.8 continues...

Compute the relative bending stiffness of all members.

\[ K_{AB} = \frac{EI}{L} = \frac{240E}{12} = 20E \]
\[ K_{BC} = \frac{EI}{L} = \frac{600E}{15} = 40E \]
\[ K_{CD} = \frac{EI}{L} = \frac{360E}{18} = 20E \]

If we set \( 20E = K \), then

\[ K_{AB} = K \quad K_{BC} = 2K \quad K_{CD} = K \] (2)

Express member end moments in terms of displacements with slope-deflection equation 12.16:

\[ M_{NF} = (2EI/L)(2\phi_N + \theta_F - 3\phi_{NP}) + FEM_{NF} \]

Since no loads are applied to members between joints, all \( FEM_{NF} = 0 \).

\[ M_{AB} = 2K_{AB}(\theta_B + 3\phi_{AB}) \]
\[ M_{BA} = 2K_{AB}(2\theta_B - 3\phi_{AB}) \]
\[ M_{BC} = 2K_{BC}(2\theta_B + \theta_C) \]
\[ M_{CB} = 2K_{BC}(2\theta_C + \theta_B) \]
\[ M_{CD} = 2K_{CD}(2\theta_C - 3\phi_{CD}) \]
\[ M_{DC} = 2K_{CD}(\theta_C - 3\phi_{CD}) \] (3)

In the equations above, use Equations 1 to express \( \phi_{AB} \) in terms of \( \phi_{CD} \), and use Equations 2 to express all stiffness in terms of the parameter \( K \).

\[ M_{AB} = 2K(\theta_B - 4.5\phi_{CD}) \]
\[ M_{BA} = 2K(2\theta_B - 4.5\phi_{CD}) \]
\[ M_{BC} = 4K(2\theta_B + \theta_C) \]
\[ M_{CB} = 4K(2\theta_C + \theta_B) \] (4)
\[ M_{CD} = 2K(2\theta_C - 3\phi_{CD}) \]
\[ M_{DC} = 2K(\theta_C - 3\phi_{CD}) \]

The equilibrium equations are:

Joint B:

\[ M_{BA} + M_{BC} = 0 \] (5)

Joint C:

\[ M_{CB} + M_{CD} = 0 \] (6)

Shear equation (see Eq. 12.21):

\[ \frac{M_{BA} + M_{AB}}{12} + \frac{M_{CD} + M_{DC}}{18} + 6 = 0 \] (7)
Substitute Equations 4 into Equations 5, 6, and 7 and combine terms.

\[
12\theta_B + 4\theta_C - 9\psi_{CD} = 0 \quad (5a)
\]
\[
4\theta_B + 12\theta_C - 6\psi_{CD} = 0 \quad (6a)
\]
\[
9\theta_B + 6\theta_C - 39\psi_{CD} = -\frac{108}{K} \quad (7a)
\]

Solving the equations above simultaneously gives

\[
\theta_B = \frac{2.257}{K}, \quad \theta_C = \frac{0.97}{K}, \quad \psi_{CD} = \frac{3.44}{K}
\]

Also,

\[
\psi_{AB} = 1.5\psi_{CD} = \frac{5.16}{K}
\]

Since all angles are positive, all joint rotations and the sidesway angles are clockwise.

Substituting the values of displacement above into Equations 4, we establish the member end moments.

\[
M_{AB} = -26.45 \text{ kip·ft} \quad M_{BA} = -21.84 \text{ kip·ft} \quad \text{Ans.}
\]
\[
M_{BC} = 21.84 \text{ kip·ft} \quad M_{CB} = 16.78 \text{ kip·ft}
\]
\[
M_{CD} = -16.76 \text{ kip·ft} \quad M_{DC} = -18.7 \text{ kip·ft}
\]

The final results are summarized in Figure 12.15b.

---

Analyze the frame in Figure 12.16a by the slope-deflection method. Given: \(EI\) is constant for all members.

**Solution**

Identify the unknown displacements; \(\theta_B, \theta_C\), and \(\psi_{AB}\). Since the cantilever is a determinate component of the structure, its analysis does not have to be included in the slope-deflection formulation. Instead, we consider the cantilever a device to apply a vertical load of 6 kips and a clockwise moment of 24 kip·ft to joint C.

Express member end moments in terms of displacements with Equation 12.16 (all units in kip·feet).

\[
M_{AB} = \frac{2EI}{8} (\theta_B - 3\psi_{AB}) - \frac{3(8)^2}{12}
\]
\[
M_{BA} = \frac{2EI}{8} (2\theta_B - 3\psi_{AB}) + \frac{3(8)^2}{12} \quad (1)^*
\]

*Two additional equations for \(M_{BC}\) and \(M_{CB}\) on page 468. [continues on next page]*
Example 12.9 continues...

Figure 12.16: (a) Details of frame: rotation of chord $\psi_{AB}$ shown by dashed line; (b) moments acting on joint $B$ (shear and axial forces omitted for clarity); (c) moments acting on joint $C$ (shear forces and reaction omitted for clarity); (d) free body of column $AB$; (e) free body of girder used to establish third equilibrium equation.

\[
\begin{align*}
M_{BC} &= \frac{2EI}{12} (2\theta_B + \theta_C) \\
M_{CB} &= \frac{2EI}{12} (2\theta_C + \theta_B)
\end{align*}
\]

Write the joint equilibrium equations at $B$ and $C$.

Joint $B$ (see Fig. 12.16b):

\[ + \sum M_B = 0: \quad M_{BA} + M_{BC} = 0 \quad (2) \]

Joint $C$ (see Fig. 12.16c):

\[ + \sum M_C = 0: \quad M_{CB} - 24 = 0 \quad (3) \]

Shear equation (see Fig. 12.16d):

\[ C^+ \sum M_A = 0 \quad M_{BA} + M_{AB} + 24(4) - V_1(8) = 0 \]

solving for $V_1$ gives

\[ V_1 = \frac{M_{BA} + M_{AB} + 96}{8} \quad (4a) \]

Isolate the girder (see Fig. 12.16e) and consider equilibrium in the horizontal direction.

\[ \rightarrow + \sum F_x = 0: \quad \text{therefore} \quad V_1 = 0 \quad (4b) \]
Substitute Equation 4a into Equation 4b:

\[ M_{BA} + M_{AB} + 96 = 0 \]  

(4)

Express equilibrium equations in terms of displacements by substituting Equations 1 into Equations 2, 3, and 4. Collecting terms and simplifying, we find

\[ 10\theta_B - 2\theta_c - 9\psi_{AB} = \frac{-192}{EI} \]
\[ \theta_B - 2\theta_c = \frac{144}{EI} \]
\[ 3\theta_B - 6\psi_{AB} = \frac{-384}{EI} \]

Solution of the equations above gives

\[ \theta_B = \frac{53.33}{EI}, \quad \theta_c = \frac{45.33}{EI}, \quad \psi_{AB} = \frac{90.66}{EI} \]

Establish the values of member end moments by substituting the values of \( \theta_B, \theta_c, \) and \( \psi_{AB} \) into Equations 1.

\[ M_{AB} = \frac{2EI}{8} \left( \frac{53.33}{EI} \cdot \frac{(3)(90.66)}{EI} \right) + 16 = -70.67 \text{ kip-ft} \]
\[ M_{BA} = \frac{2EI}{8} \left( \frac{(2)(53.33)}{EI} - \frac{(3)(90.66)}{EI} \right) - 16 = -25.33 \text{ kip-ft} \]
\[ M_{BC} = \frac{2EI}{12} \left( \frac{(2)(53.33)}{EI} + \frac{45.33}{EI} \right) = 25.33 \text{ kip-ft} \]
\[ M_{CB} = \frac{2EI}{12} \left( \frac{(2)(45.33)}{EI} + \frac{53.33}{EI} \right) = 24 \text{ kip-ft} \]

After the end moments are established, we compute the shears in all members by applying the equations of equilibrium to free bodies of each member. Final results are shown in Figure 12.16f.

Figure 12.16: (f) Reactions and shear and moment curves.
Analyze the frame in Figure 12.17a by the slope-deflection method. Determine the reactions, draw the moment curves for the members, and sketch the deflected shape. If $I = 240 \text{ in}^4$ and $E = 30,000 \text{ kips/in}^2$, determine the horizontal displacement of joint $B$.

**Solution**

Unknown displacements are $\theta_b, \theta_c, \text{and} \psi$. Since supports at $A$ are fixed, $\theta_A$ and $\theta_D$ equal zero. There is no chord rotation of girder $BC$.

Express member end moments in terms of displacements with the slope-deflection equation. Use Figure 12.5 to evaluate $\text{FEM}_{NF}$.

$$M_{NF} = \frac{2EI}{L} (2\theta_N + \theta_P - 3\psi_{NF}) + \text{FEM}_{NF} \quad (12.16)$$

$$\text{FEM}_{BC} = -\frac{Pb^2a}{L^2} = \frac{12(30)^2(15)}{(45)^2}$$

$$\text{FEM}_{CD} = \frac{Pa^2b}{L^2} = \frac{12(15)^2(30)}{(45)^2}$$

$$= -80 \text{ kip-ft}$$

To simplify slope-deflection expressions, set $EI/15 = K$.

$$M_{AB} = \frac{2EI}{15} (\theta_B - 3\psi) = 2K(\theta_B - 3\psi)$$

$$M_{BA} = \frac{2EI}{15} (2\theta_B - 3\psi) = 2K(2\theta_B - 3\psi)$$

$$M_{BC} = \frac{2EI}{45} (2\theta_B + \theta_C) - 80 = \frac{2}{3}K(2\theta_B + \theta_C) - 80$$

$$M_{CB} = \frac{2EI}{45} (2\theta_C + \theta_B) + 40 = \frac{2}{3}K(2\theta_B + \theta_C) + 40$$

$$M_{CD} = \frac{2EI}{15} (2\theta_C - 3\psi) = 2K(\theta_C - 3\psi)$$

The equilibrium equations are:

Joint $B$: $M_{BA} + M_{BC} = 0 \quad (2)$

Joint $C$: $M_{CB} + M_{CD} = 0 \quad (3)$

Shear equation (see the girder in Fig. 12.17b):

$$\sum F_x = 0 \quad V_1 + V_2 = 0 \quad (4a)$$
where \( V_1 = \frac{M_{BA} + M_{AB}}{15} \) \( V_2 = \frac{M_{CD} + M_{DC}}{15} \) (4b)

Substituting \( V_1 \) and \( V_2 \) given by Equations 4b into 4a gives

\[
M_{BA} + M_{AB} + M_{CD} + M_{DC} = 0
\]

(4)

Alternatively, we can set \( Q = 0 \) in Equation 12.21 to produce Equation 4.

Express equilibrium equations in terms of displacements by substituting Equations 1 into Equations 2, 3, and 4. Combining terms and simplifying give

\[
8K\theta_B + K\theta_C - 9K\psi = 120
\]

\[
2K\theta_B + 16K\theta_C - 3K\psi = -120
\]

\[
K\theta_B + K\theta_C - 4K\psi = 0
\]

Solving the equations above simultaneously, we compute

\[
\theta_B = \frac{410}{21K} \quad \theta_C = -\frac{130}{21K} \quad \psi = \frac{10}{3K}
\]

(5)

Substituting the values of the \( \theta_B, \theta_C, \) and \( \psi \) into Equations 1, we compute the member end moments below.

\[
M_{AB} = 19.05 \text{ kip-ft} \quad M_{BA} = 58.1 \text{ kip-ft}
\]

\[
M_{CD} = -44.76 \text{ kip-ft} \quad M_{DC} = -32.38 \text{ kip-ft}
\]

(6)

\[
M_{BC} = -58.1 \text{ kip-ft} \quad M_{CB} = 44.76 \text{ kip-ft}
\]

Member end moments and moment curves are shown on the sketch in Figure 12.17c; the deflected shape is shown in Figure 12.17d.

\[\text{continues on next page}\]
Example 12.10 continues... Compute the horizontal displacement of joint $B$. Use Equation 1 for $M_{AB}$. Express all variables in units of inches and kips.

$$M_{AB} = \frac{2EI}{15(12)} (\theta - 3\psi) \quad (7)$$

From the values in Equation 5 (p. 485), $\theta = 5.86\psi$; substituting into Equation 7, we compute

$$19.05(12) = \frac{2(30,000)(240)}{15(12)} (5.86\psi - 3\psi)$$

$$\psi = 0.000999 \text{ rad}$$

$$\psi = \frac{\Delta}{L} \quad \Delta = \psi L = 0.000999(15 \times 12) = 0.18 \text{ in} \quad \text{Ans.}$$

12.6 Kinematic Indeterminacy

To analyze a structure by the flexibility method, we first established the degree of indeterminacy of the structure. The degree of statical indeterminacy determines the number of compatibility equations we must write to evaluate the redundants, which are the unknowns in the compatibility equations.

In the slope-deflection method, displacements—both joint rotations and translations—are the unknowns. As a basic step in this method, we must write equilibrium equations equal in number to the independent joint displacements. The number of independent joint displacements is termed the degree of kinematic indeterminacy. To determine the kinematic indeterminacy, we simply count the number of independent joint displacements that are free to occur. For example, if we neglect axial deformations, the beam in Figure 12.18a is kinematically indeterminate to the first degree. If we were to analyze this beam by slope-deflection, only the rotation of joint $B$ would be treated as an unknown.

If we also wished to consider axial stiffness in a more general stiffness analysis, the axial displacement at $B$ would be considered an additional unknown, and the structure would be classified as kinematically indeterminate to the second degree. Unless otherwise noted, we will neglect axial deformations in this discussion.

In Figure 12.18b the frame would be classified as kinematically indeterminate to the fourth degree because joints $A$, $B$, and $C$ are free to rotate.
and the girder can translate laterally. Although the number of joint rotations is simple to identify, in certain types of problems the number of independent joint displacements may be more difficult to establish. One method to determine the number of independent joint displacements is to introduce imaginary rollers as joint restraints. The number of rollers required to restrain the joints of the structure from translating equals the number of independent joint displacements. For example, in Figure 12.18c the structure would be classified as kinematically indeterminate to the eighth degree, because six joint rotations and two joint displacements are possible. Each imaginary roller (noted by the numbers 1 and 2) introduced at a floor prevents all joints in that floor from displacing laterally. In Figure 12.18d the Vierendeel truss would be classified as kinematically indeterminate to the eleventh degree (i.e., eight joint rotations and three independent joint translations). Imaginary rollers (labeled 1, 2, and 3) added at joints B, C, and H prevent all joints from translating.

**Summary**

- The slope-deflection procedure is an early classical method for analyzing indeterminate beams and rigid frames. In this method joint displacements are the unknowns.
- For highly indeterminate structures with a large number of joints, the slope-deflection solution requires that the engineer solve a series of simultaneous equations equal in number to the unknown displacements—a time-consuming operation. While the use of the slope-deflection method to analyze structures is impractical given the availability of computer programs, familiarity with the method provides students with valuable insight into the behavior of structures.
- As an alternate to the slope-deflection method, moment distribution was developed in the 1920s to analyze indeterminate beams and frames by distributing unbalanced moments at joints in an artificially restrained structure. While this method eliminates the solution of simultaneous equations, it is still relatively long, especially if a large number of loading conditions must be considered. Nevertheless, moment distribution is a useful tool as an approximate method of analysis both for checking the results of a computer analysis and in making preliminary studies. We will use the slope-deflection equation (in Chap. 13) to develop the moment distribution method.
- A variation of the slope-deflection procedure, the general stiffness method, used to prepare general-purpose computer programs, is presented in Chapter 16. This method utilizes stiffness coefficients—forces produced by unit displacements of joints.

![Diagram](image-url)
PROBLEMS


P12.3. Analyze by slope-deflection and draw the shear and moment curves for the beam in Figure P12.3. Given: $EI = $constant.

P12.4. Analyze the beam in Figure P12.4 by slope-deflection and draw the shear and moment diagrams for the beam. $EI$ is constant.

P12.5. Analyze by slope-deflection and draw the shear and moment curves for the continuous beam in Figure P12.5. Given: $EI$ is constant.

P12.6. Draw the shear and moment curves for the frame in Figure P12.6. Given: $EI$ is constant. How does this problem differ from Problem P12.5?
P12.7. Compute the reactions at A and C in Figure P12.7. Draw the shear and moment diagram for member BC. Given: $I = 2000 \text{ in}^4$ and $E = 3,000 \text{ kips/in}^2$.

P12.9. (a) Under the applied loads support B in Figure P12.9 settles 0.5 in. Determine all reactions. Given: $E = 30,000 \text{ kips/in}^2$, $I = 240 \text{ in}^4$. (b) Compute the deflection of point C.

P12.8. Use the slope-deflection method to determine the vertical deflection at B and the member end moments at A and B for the beam in Figure P12.8. $EI$ is a constant. The guide support at B permits vertical displacement, but allows no rotation or horizontal displacement of the end of the beam.

P12.10. In Figure P12.10, support A rotates 0.002 rad and support C settles 0.6 in. Draw the shear and moment curves. Given: $I = 144 \text{ in}^4$ and $E = 29,000 \text{ kips/in}^2$. 
In Problems P12.11 to P12.14, take advantage of symmetry to simplify the analysis by slope deflection.

P12.11. (a) Compute all reactions and draw the shear and moment curves for the beam in Figure P12.11. Given: $EI$ is constant. (b) Compute the deflection under the load.

P12.12. (a) Determine the member end moments for the rectangular ring in Figure P12.12, and draw the shear and moment curves for members $AB$ and $AD$. The cross section of the rectangular ring is 12 in $\times$ 8 in and $E = 3000$ kips/in$^2$. (b) What is the axial force in member $AD$ and in member $AB$?

P12.13. Figure P12.13 shows the forces exerted by the soil pressure on a typical 1-ft length of a concrete tunnel as well as the design load acting on the top slab. Assume a fixed-end condition at the bottom of the walls at $A$ and $D$ is produced by the connection to the foundation mat. $EI$ is constant.

P12.14. Compute the reactions and draw the shear and moment curves for the beam in Figure P12.14. Also $E = 200$ GPa and $I = 120 \times 10^6$ mm$^4$. Use symmetry to simplify the analysis. Fixed ends at supports $A$ and $E$. 
**P12.15.** Consider the beam in Figure P12.14 without the applied load. Compute the reactions and draw the shear and moment curves for the beam if support C settles 24 mm and support A rotates counterclockwise 0.005 rad.

**P12.16.** Analyze the frame in Figure P12.16. Given: $EI$ is constant for all members. Use symmetry to simplify the analysis.

**P12.17.** Analyze the frame in Figure P12.17. Given: $EI$ is constant. Fixed ends at $A$ and $D$.

**P12.18.** Analyze the structure in Figure P12.18. In addition to the applied load, support $A$ rotates clockwise by 0.005 rad. Also $E = 200$ GPa and $I = 25 \times 10^6$ mm$^4$ for all members. Fixed end at $A$.

**P12.19.** Analyze the frame in Figure P12.19. Given: $EI$ is constant. Fixed supports at $A$ and $B$. 
P12.20. (a) Draw the shear and moment curves for the frame in Figure P12.20. (b) Compute the deflection at midpoint of girder BC. Given: $E = 29,000$ kips/in$^2$.

P12.21. Analyze the frame in Figure P12.21. Compute all reactions. Also $I_{BC} = 200$ in$^4$ and $I_{AB} = I_{CD} = 150$ in$^4$. $E$ is constant.

P12.22. Analyze the frame in Figure P12.22. Also $EI$ is constant. Notice that sidesway is possible because the load is unsymmetric. Compute the horizontal displacement of joint B. Given: $E = 29,000$ kips/in$^2$ and $I = 240$ in$^4$ for all members.

P12.23. Compute the reactions and draw the shear and moment diagrams for beam BC in Figure P12.23. Also $EI$ is constant.
P12.24. Determine all reactions in Figure P12.24. Draw the shear and moment diagrams for member BC. The ends of the beams at points A and C are embedded in concrete walls that produce fixed supports. The light baseplate at D may be treated as a pin support. Also EI is constant.

P12.25. Determine all reactions at points A and D in Figure P12.25. EI is constant.

P12.26. If support A in Figure P12.26 is constructed 0.48 in too low and the support at C is accidentally constructed at a slope of 0.016 rad clockwise from a vertical axis through C, determine the moment and reactions created when the structure is connected to its supports. Given: $E = 29,000$ kips/in$^2$.

P12.27. If member AB in Figure P12.27 is fabricated $\frac{3}{4}$ in too long, determine the moments and reactions created in the frame when it is erected. Sketch the deflected shape. $E = 29,000$ kips/in$^2$. 

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**Problems**
P12.28. Set up the equilibrium equations required to analyze the frame in Figure P12.28 by slope deflection. Express the equilibrium equations in terms of the appropriate displacements; $EI$ is constant for all members.

![Figure P12.28](image)

P12.29. Analyze the frame in Figure P12.29. Also $EI$ is constant. Fixed supports at $A$ and $D$.

![Figure P12.29](image)
P12.30. Determine the degree of kinematic indeterminacy for each structure in Figure P12.30. Neglect axial deformations.
East Bay Drive, a post-tensioned concrete frame bridge, 146 ft long,.mainspan 60 ft, edge of concrete
girder 7 in thick.