CHAPTER FOUR

4. SLOPE – DEFLECTION METHOD

This method is applicable to all types of statically indeterminate beams & frames and in this method, we solve for unknown joint rotations, which are expressed in terms of the applied loads and the bending moments. By inspection, the degree of indeterminacy is checked and the corresponding number of unknown joint rotations are calculated from the slope – deflections equations.

4.1. SIGN CONVENTION:–

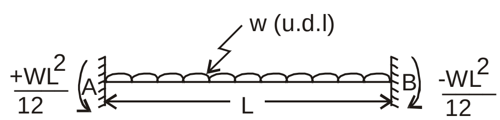
(1) ROTATIONS:– Clockwise joint rotations are considered as (+ve).

(2) END MOMENTS:– Counterclockwise end moments are considered as (+ve).

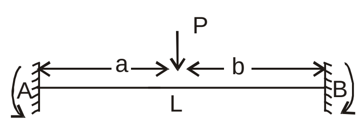
4.2. PROCEDURE:–

The procedure is as follows:

(1) Determine the fixed end moments at the end of each span due to applied loads acting on span by considering each span as fixed ended. Assign ± Signs w.r.t. above sign convention.



$$Mf_{ab} = \frac{+WL^2}{12} \qquad MF_{ba} = \frac{-WL^2}{12}$$



$$Mf_{ab} = \frac{+Pab^2}{L^2} \qquad MF_{ba} = \frac{-Pa^2b}{L^2}$$

(2) Express all end moments in terms of fixed end moments and the joint rotations by using slope – deflection equations.

(3) Establish simultaneous equations with the joint rotations as the unknowns by applying the condition that sum of the end moments acting on the ends of the two members meeting at a joint should be equal to zero.

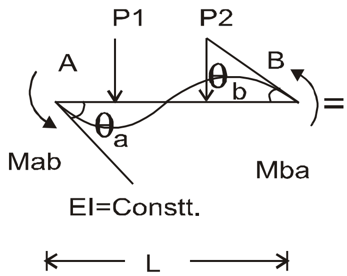
(4) Solve for unknown joint rotations.

(5) Substitute back the end rotations in slope – deflection equations and compute the end moments.

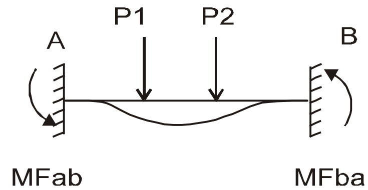
(6) Determine all reactions and draw S.F. and B.M. diagrams and also sketch the elastic curve

4.3. DERIVATION OF SLOPE – DEFLECTION EQUATION:–

Consider a generalized beam under the action of applied loads and end moments as shown at (i).

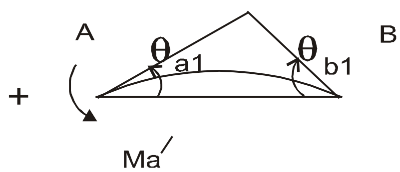


(i)

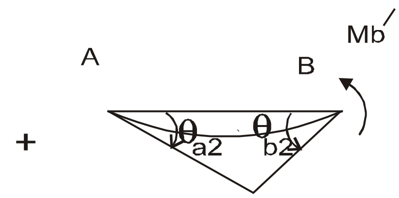


(ii)

Fig: (i) can be equated to a fixed ended beam carrying applied loads which produce fixing moments plus two simple beams carrying end moments [figs (iii) and (iv)]

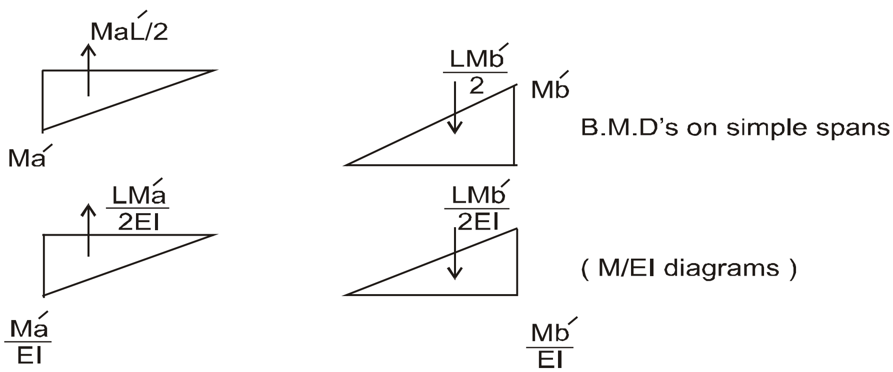


(iii)



(iv)

Draw moment diagrams. Determine their areas and centroid locations.



(Assuming these $\frac{M}{EI}$ diagrams are placed on conjugate beams)

Equating relevant rotations in above four diagrams according to sign conventions

$$\theta_a = 0 - \theta a_1 + \theta a_2 = -\theta a_1 + \theta a_2$$

and $\theta_b = 0 + \theta b_1 - \theta b_2 = \theta b_1 - \theta b_2$ (1) Compatibility on rotations

During the same for moments.

So Mab = Mfab + Ma′

Mba = Mfba + Mb′ (2) Compatibility on moments

Where Ma′ and Mb′ are the additional moments required to produce the joint rotations at ends A and B respectively and Mfab & Mfba are the fixed ended moments which hold the tangents at points A and B straight.Conjugate beam theorem states that " rotation at a point in actual beam is equal to the shear force at the corresponding point in the conjugate beam). Applying it we have.

$$\theta a_1 = \frac{2}{3}\left(\frac{LMa'}{2EI}\right) = \frac{LMa'}{3EI}$$

$$\theta b_1 = \frac{1}{3}\left(\frac{LMa'}{2EI}\right) = \frac{LMa'}{6EI}$$

$$\theta a_2 = \frac{1}{3}\left(\frac{LMb'}{2EI}\right) = \frac{LMb'}{6EI}$$

$$\theta b_2 = \frac{2}{3}\left(\frac{LMb'}{2EI}\right) = \frac{LMb'}{3EI}$$

Putting the values of θa1, θa2, θb1 & θb2 in equation (1) and solve for Ma′ & Mb′.

$$\theta a = -\frac{LMa'}{3EI} + \frac{LMb'}{6EI} = -\frac{L}{3}\frac{Ma'}{EI} + \frac{LMb'}{6EI} \quad \rightarrow (3)$$

and $$\theta b = \frac{Ma'L}{6EI} - \frac{LMb'}{3EI} = \frac{L}{6}\frac{Ma'}{EI} - \frac{L}{3}\frac{Mb'}{EI} \quad \rightarrow (4)$$

Equation (3) becomes $\theta a + \frac{LMa'}{3EI} = \frac{LMb'}{6EI}$ OR

$$\frac{6EI\theta a + 2LMa'}{6EI} = \frac{LMb'}{6EI}$$

$$6EI\theta a + 2\,LMa' = LMb'$$

$$Mb' = \frac{6EI}{L}\theta a + 2\,Ma' \quad \rightarrow (5)$$

From (4), $\theta b = \frac{Ma'L}{6EI} - \frac{L}{3EI}\left(\frac{6EI\theta a}{L} + 2Ma'\right)$ by putting Mb' from (5)

$$\theta b = \frac{Ma'L}{6EI} - 2\,\theta a - \frac{2LMa'}{3EI}$$

$$\theta b + 2\,\theta a = \frac{Ma'L}{6EI} - \frac{2LMa'}{3EI}$$

$$\theta b + 2\theta a = \frac{Ma'L - 4\,LMa'}{6EI}$$

$$\theta b + 2\theta a = \frac{-3LMa'}{6EI}$$

So $\theta b + 2\theta a = \frac{-LMa'}{2EI}$ From here Ma' is

$$Ma' = \frac{-2EI}{L}(2\theta a + \theta b)$$

or $Ma' = \frac{2EI}{L}(-2\theta a - \theta b)$ $\rightarrow$ (6)

From(5) $Mb' = \frac{6EI\,\theta a}{L} + \frac{4EI}{L}(-2\,\theta a - \theta b)$ By putting value of Ma' from 6 in 5 and simplifying

$$Mb' = \frac{6EI\,\theta a}{L} - \frac{8EI\,\theta a}{L} - \frac{4EI}{L}\theta b$$

$$Mb' = \frac{-2EI\,\theta a}{L} - \frac{4EI}{L}\theta b$$

or $Mb' = \frac{2EI}{L}(-\theta a - 2\,\theta b)$ $\rightarrow$ (7)

Putting the values of Ma′ and Mb′ from equations 6 and 7 in equation (2), we have.

$$Mab = Mfab + \frac{2EI}{L}(-2\theta a - \theta b)$$
$$Mba = Mfba + \frac{2EI}{L}(-\theta a - 2\theta b)$$

Absolute values of $\frac{2EI}{L}$ are not required in general except for special cases and we use relative values of $\frac{2EI}{L}$ in cases without settlement..

Where, $K = \frac{I}{L}$ if absolute stiffness (rotation) is not required.

Where K = relative stiffness

Slope – deflection equation for members without settlement.

$$Mab = Mfab + \frac{2EI}{L}(-2\theta a - \theta b)$$

$$Mba = Mfba + \frac{2EI}{L}(-2\,\theta b - \theta a)$$

without absolute value of $\frac{2EI}{L}$, above equations become

$$Mab = Mfab + Kab(-2\theta a - \theta b)$$

$$Mba = Mfba + Kab(-2\theta b - \theta a)$$

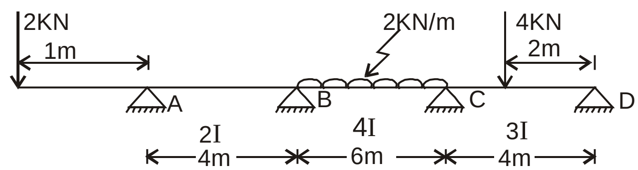
Where Kab = relative stiffness of member ab

$$Kab = \left(\frac{2EI}{L}\right)^{ab}$$

Now we apply the method to various indeterminate structures.

EXAMPLE NO.1::– Analyze the continuous beam shown by slope – deflection method. Draw shear & moment diagram and sketch the elastic curve.

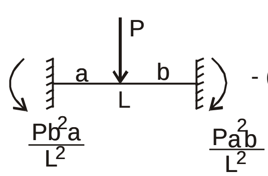
SOLUTION :–



Step 1: **Calculation of Relative Stiffness** :–

Member.	I	L	$\frac{I}{L}$	Krel.
AB	2	4	$\frac{2}{4}\times 12$	6
BC	4	6	$\frac{4}{6}\times 12$	8
CD	3	4	$\frac{3}{4}\times 12$	9

Step 2: **Calculation of Fixed End Moments :–** Treat each span as fixed ended.



Mfab = Mfba = 0 (there is no load acting on span AB)

$Mfbc = \frac{2 \times 6^2}{12} = +6$ KN–m (According to our sign convention)

Mfcb = – 6 KN–m (According to our sign convention)

$$Mfcd = \frac{4 \times 2^2 \times 2}{4^2} = +2 \text{ KN–m}$$

Mfdc = – 2 KN–m

Step 3: **Establish simultaneous equations** :–

Mab = Mfab + Kab (–2 θa – θb) (General form–Put values of FEMs & relative stiffnesses)

Mab = 0 + 6 (– 2θa – θb) = – 12 θa – 6 θb

Mba = 6 (– 2θb – θa) = – 12 θb – 6 θa

Mbc = 6 + 8 (– 2 θb – θc) = 6 – 16 θb – 8 θc

Mcb = – 6 + 8 (– 2 θc – θb) = – 6 – 16 θc – 8 θb

Mcd = 2 + 9 (– 2θc – θd) = 2 – 18 θc – 9 θd

Mdc = – 2 + 9 (–2θd – θc) = – 2 – 18 θd – 9 θc

Step 4: **Joint Conditions :–**

at A : Mab – 2=0 or Mab = 2 KN–m

B Mba + Mbc = 0

C: Mcb + Mcd = 0

D: Mdc = 0

Put these joint conditions in the linear simultaneous equations set up in step No. (3).

Mab = 2, so – 12 θa – 6 θb = 2

– 12 θa – 6 θb – 2 = 0 → (1)

Mba + Mbc = 0

so – 12 θb – 6 θa + 6 – 16 θb – 8 θc = 0

– 6 θa – 28 θb – 8 θc + 6 = 0 → (2)

Mcb + Mcd = 0

so – 6 – 16θc – 8 θb + 2 – 18 θc – 9 θd = 0

– 8θb – 34 θc – 9 θd – 4 = 0 → (3)

Mdc = 0

– 2 – 18 θd – 9 θc = 0

– 9 θc – 18 θd – 2 = 0 → (4)

– 12 θa – 6 θb – 2 = 0
– 6 θa – 28 θb – 8 θc + 6 = 0 (Symmetrical about θa and θd diagonal)
0 – 8 θb – 34 θc – 9 θd – 4 = 0
0 – 0 – 9 θc – 18 θd– 2 = 0

If the linear simultaneous equations are established and are arranged in a sequence of joint conditions, we will find that the quantities on the leading diagonal are dominant in that particular equation and off diagonal quantities are symmetrical as far as the magnitude of rotations is concerned. This is a typical property of the stiffness method, which you will study later in matrix methods of structural analysis.

From (1) $\theta a = \left(\frac{-2 - 6\theta b}{12}\right) \rightarrow (5)$

From (4) $\theta d = \left(\frac{-2 - 9\,\theta c}{18}\right) \rightarrow (6)$

Putting these values in equations (2) & (3), all deformations are expressed in terms of θb & θc. Therefore, we get two linear simultaneous equations in terms of θb & θc. Hence, their values can be calculated.

Put θa from equations (5) in equation (2)

$$-6\left(\frac{-2 - 6\theta b}{12}\right) - 28\,\theta b - 8\,\theta c + 6 = 0$$

$$+1 + 3\,\theta b - 28\,\theta b - 8\,\theta c + 6 = 0$$

or $-25\,\theta b - 8\,\theta c + 7 = 0 \rightarrow (7)$

Put θd from equation (6) in equation (3)

$$-8\,\theta b - 34\,\theta c - 9\left(\frac{-2 - 9\,\theta c}{18}\right) - 4 = 0 \qquad \text{Simplifying}$$

$$-8\,\theta b - 34\,\theta c + 1 + 4.5\,\theta c - 4 = 0$$

$$-8\,\theta b - 29.5\,\theta c - 3 = 0 \rightarrow (8)$$

From (7) $\theta b = \left(\frac{-8\,\theta c + 7}{25}\right) \rightarrow (9)$

Put in (8) $-8\left(\frac{-8\,\theta c + 7}{25}\right) - 29.5\,\theta c - 3 = 0$

or $2.56\,\theta c - 2.24 - 29.5\,\theta c - 3 = 0$

$$-26.94\,\theta c - 5.24 = 0$$

$$\theta c = \frac{-5.24}{26.94}$$

$$\boxed{\theta c = -0.1945} \quad \text{Radians}$$

Put value of θc in equation (9) , we get

$$\theta b = \left[\frac{-8\,(-0.1945) + 7}{25}\right]$$

$$\boxed{\theta b = +\,0.3422} \qquad \text{radians.}$$

Put θb in equation (5)

$$\theta a = \left(\frac{-2 - 6 \times 0.3422}{12}\right)$$

$$\boxed{\theta a = -\,0.3378} \qquad \text{radians.}$$

Put θc in equation (6)

$$\theta d = \frac{-2 - 9\,.\,(-\,0.1945)}{18}$$

$$\boxed{\theta d = -\,0.0139} \qquad \text{radians.}$$

Putting these values of rotations in simultaneous equations set up in step (3) & simplifying we get the values of end moments as under:

Mab = 2 KN–m

Mba = – 2.08 KN–m
Mbc = + 2.08 KN–m

These two values should be the the same but with opposite signs to satisfy equilibrium at that joint.

Mcb = – 5.63 KN–m
Mcd = + 5.63 KN–m

(Same comment)

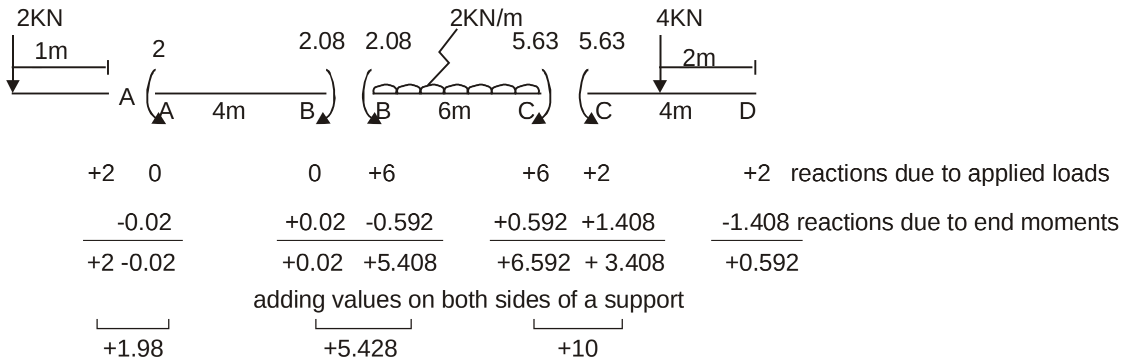
Mdc = 0

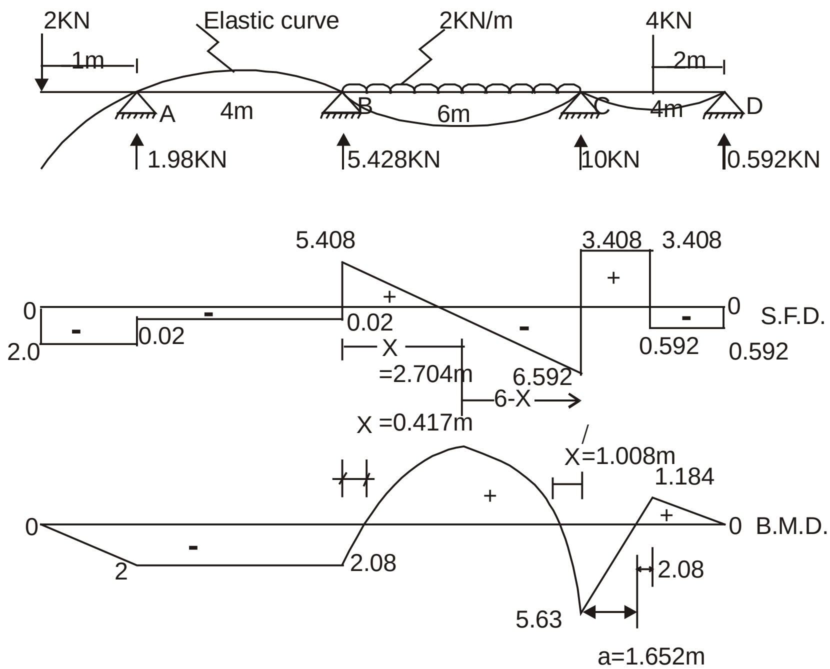
As the end moments have been calculated and they also satisfy the joint conditions, therefore, the structure is statically determinate at this stage. Reactions, shear force diagrams, B.M. diagrams & elastic curves can now be sketched.

NOTE:–

In slope – deflection method, the actual deformations are the redundants and stiffness matrix is symmetrical. In force – method, we can chose any redundant and therefore flexibility matrix is not generally symmetrical about leading diagonal.

Now we can draw shear force and bending moment diagrams and sketch elastic curve. Free body diagrams of various spans are drawn.





Find the location of points of contraflexure & find the maximum +ve B. M. in portion BC by setting the relevant moment expression equal to zero and by setting the concerned S.F. expression equal to zero respectively.

To Find Max B.M. in Portion BC :–

$$\frac{X}{5.408} = \frac{6 - X}{6.592}$$

$$6.592\,X = 6 \times 5.408 - 5.408\,X$$

$$X = 2.704\text{m}$$

So $$Mbc = -2.08 + 5.408 \times 2.704 - \frac{2}{2} \times (2.704)$$

$$Mbc = 5.237 \text{ KN–m}$$

Points of Contraflexure :–

Near B:–

$$-2.08 + 5.408 X - X^2 = 0$$

$$X^2 - 5.408 X + 2.08 = 0$$

$$X = \frac{5.408 \pm \sqrt{(5.408)^2 - 4 \times 1 \times 2.08}}{2 \times 1}$$

$$X = 0.417 \text{ m}, \ 4.991 \text{ m}$$

$$\boxed{X = 0.417 \text{ m}}$$

Near C :– In span CB

$$-5.63 + 6.592 X' - X'^2 = 0$$

$$X'^2 - 6.592 X' + 5.63 = 0$$

$$X' = \frac{6.592 \pm \sqrt{(6.592)^2 - 4 \times 1 \times 5.63}}{21}$$

$$X' = \frac{6.592 \pm 4.575}{2}$$

$$X' = 5.584, \ 1.008$$

$$\boxed{X' = 1.008 \text{ m}}$$

$$\frac{1.184}{2 - a} = \frac{5.63}{a} \quad \text{in span CD.}$$

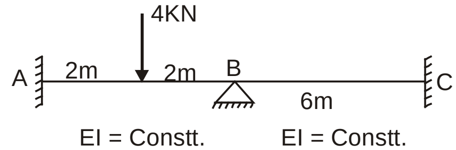
$$1.184 a = 5.63 \times 2 - 5.63 a$$

$$\boxed{a = 1.652 \text{ m}}$$

These can be put in bending moment diagram and sketch elastic curve.

EXAMPLE NO. 2:– Analyse the continuous beam shown by slope –deflection method. Draw S.F.D. & B.M.D. Also sketch the elastic curve.

SOLUTION :–



Step 1: **Calculation of Relative Stiffness :–**

Member	I	L	$\frac{I}{L}$	Krel.
AB	1	4	$\frac{1}{4} \times 12$	3
BC	1	6	$\frac{1}{6} \times 12$	2

Step 2: **Calculation of Fixed End Moments** :– Mab = Mfab + Krel (–2ba – θb)

$$Mfab = \frac{4 \times 2^2 \times 2}{4^2} = + 2 \text{ KN–m}$$

Mfba = – 2 KN–m

Mfbc = 0

Mfcb = 0 (As there is no load in portion BC)

Step 3: **Establish Simultaneous Equations** :–

Mab = 2 + 3 (– 2 θa – θb)

Mba = – 2 + 3 (– 2 θb – θa)

Mbc = 0 + 2 (– 2 θb – θc)

Mcb = 0 + 2 (– 2 θc – θb)

Step 4: **Joint Conditions** :–

A : θa = 0 (Being a fixed joint)

B : Mba + Mbc = 0

C: θc = 0 (Being a fixed end)

Putting these joint conditions in the linear simultaneous equations set up in step No. (3)

Put θa = θc = 0 in above equations. The only equation is obtained from joint B. That becomes.

– 2 – 6 θb – 3 θa – 4 θb – 2 θc = 0

– 2 – 6 θb – 0 – 4 θb – 0 = 0

– 2 – 10 θb = 0

$\boxed{\theta b = -0.2}$ radians.

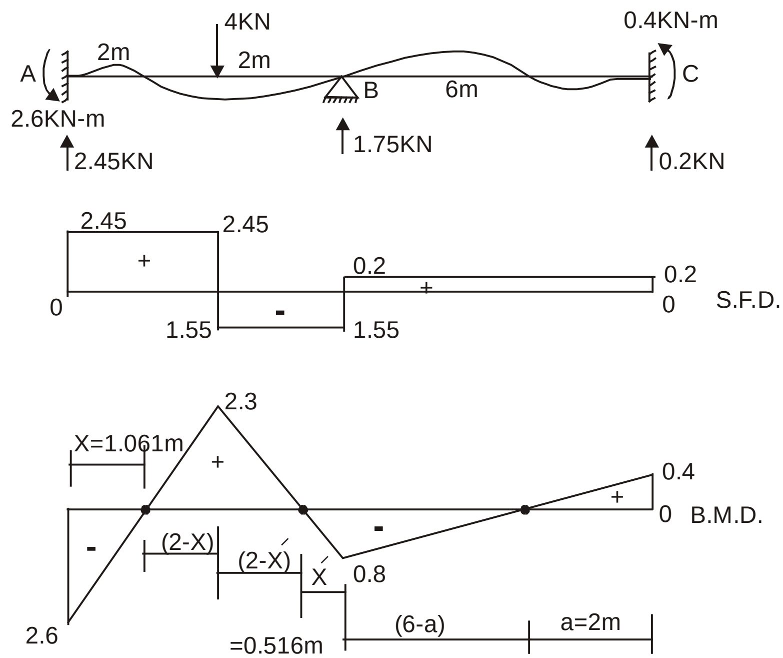
Put these values of rotations i.e., θa = θc = 0 and θb = –0.2 in simultaneous equations set up in step (3) & get the values of end moments.

Mab = 2 + 3 (– 2 × 0 + 0.2) = 2.6 KN–m

Mba = – 2 + 3 (– 2 × (– 0.2) – 0) = – 0.8 KN–m.

Mbc = 0 + 2 [– 2 × (–0.2) – 0] = + 0.8 KN–m

Mcb = 0 + 2 (0 + 0.2) = + 0.4 KN–m . Now Draw SFD and BMD.



As the end moments have been calculated and they satisfy the joint conditions, therefore, the structure is statically determinate at this stage. Reactions, S.F. diagram, B.M. diagram & elastic curve have now been sketched.

LOCATION OF POINTS OF CONTRAFLEXURE :–

Near A :–

$$\frac{2.6}{X} = \frac{2.3}{2 - X}$$

$$2.6 \times 2 - 2.6\,X = 2.3\,X$$

$$\boxed{X = 1.061\text{ m}}$$

Near B :–

$$\frac{X'}{0.8} = \frac{2 - X'}{2.3}$$

$$2.3\,X' = 2 \times 0.8 - 0.8\,X'$$

$$\boxed{X' = 0.516\text{ m}}$$

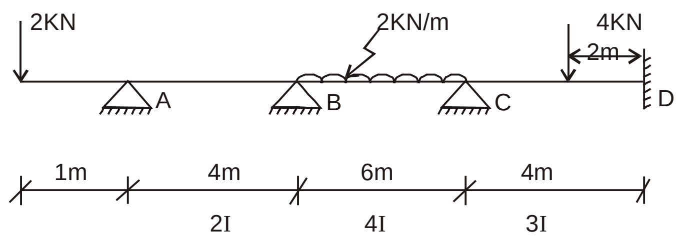
Near C :– $\frac{a}{0.4} = \left(\frac{6-a}{0.8}\right)$

$0.8\,a = 6 \times 0.4 - 0.4\,a$

$\boxed{a = 2\text{ m}}$ There have been shown on BMD.

EXAMPLE NO. 3:– Analyze the continuous beam shown by slope – deflection method. Draw S.F.D & B.M.D. Also sketch the elastic curve.

SOLUTION:–



Step 1: **Calculation of relative stiffness** :–

Member.	I	L	$\frac{I}{L}$	Krel.
AB	2	4	$\frac{2}{4} \times 12$	6
BC	4	6	$\frac{4}{6} \times 12$	8
CD	3	4	$\frac{3}{4} \times 12$	9

Step 2: **Calculation of Fixed End Moments :–**

Mfab = Mfba = 0 (no load over span AB)

Mfbc $= \frac{2 \times 6^2}{12} = +6$ KN–m

Mfcb = – 6 KN–m

Mfcd $= \frac{4 \times 2^2 \times 2}{4^2} = +2$ KN–m

Mfdc = –2 KN–m

Step 3: **Establish simultaneous equations** :– Put values of fixing moments and Krel.

$Mab = 0 + 6 (-2\ \theta a - \theta b) = -12\ \theta a - 6\ \theta b$

$Mba = 0 + 6 (-2\ \theta b - \theta a) = -12\ \theta b - 6\ \theta a$

$Mbc = 6 + 8 (-2\ \theta b - \theta c) = 6 - 16\ \theta b - 8\ \theta c$

$Mcb = -6 + 8 (-2\ \theta c - \theta b) = -6 - 16\ \theta c - 8\ \theta b$

$Mcd = 2 + 9 (-2\ \theta c - \theta d)\ 2 - 18\ \theta c - 9\ \theta d$

$Mdc = -2 + 9 (-2\ \theta d - \theta c) = -2 - 18\ \theta d - 9\ \theta c$

Step 4: **Joint Conditions** :–

A: : $Mab - 2 = 0$ or $Mab = 2$ KN–m

B : $Mba + Mbc = 0$

C : $Mcb + Mcd = 0$

D : $\theta d = 0$

Putting these joint conditions in the linear simultaneous equations set up in step No. (3)

$-12\ \theta a - 6\ \theta b = 2$ $\therefore$ $Mab = 2$

$-12\ \theta a - 6\ \theta b - 2 = 0$ → (1)

$Mba + Mbc = 0$

$-12\ \theta b - 6\ \theta a + 6 - 16\ \theta b - 8\ \theta c = 0$

$-6\ \theta a - 28\ \theta b - 8\ \theta c + 6 = 0$ → (2)

$Mcb + Mcd = 0$

$-6 - 16\ \theta c - 8\ \theta b + 2 - 18\ \theta c - 9\ \theta d = 0$

$-8\ \theta b - 34\ \theta c - 9\ \theta d - 4 = 0$ → (3)

$\boxed{\theta d = 0}$ (4) Simplifying we get.

$-12\ \theta a - 6\ \theta b - 2 = 0$ → (1)

$-6\ \theta a - 28\ \theta b - 8\ \theta c + 6 = 0$ → (2)

$-8\ \theta b - 34\ \theta c - 9\ \theta d - 4 = 0$ → (3)

$\theta d = 0$ → (4)

Putting the value of θd in equation (3)

$-8\theta b - 34\ \theta c - 0 - 4 = 0$

$-8\ \theta b - 34\ \theta c - 4 = 0$ → (5)

From (1) $\theta a = \left(\dfrac{-6\ \theta b - 2}{12}\right)$ → (6)

Put in (2) $-6\left(\dfrac{-6\ \theta b - 2}{12}\right) - 28\ \theta b - 8\ \theta c + 6 = 0$

$+3\ \theta b + 1 - 28\ \theta b - 8\ \theta c + 6 = 0$

$-25\ \theta b - 8\ \theta c + 7 = 0$ → (7)

From (5) $\theta b = \left(\dfrac{-34\ \theta c - 4}{8}\right)$ → (8)

Put in (7) $-25\left(\dfrac{-34\ \theta c - 4}{8}\right) - 8\ \theta c + 7 = 0$

or $106.25\ \theta c + 12.5 - 8\ \theta c + 7 = 0$

$98.25\ \theta c + 19.5 = 0$

$\boxed{\theta c = -0.1985}$ radians.

From (8) $\theta b = \left[\dfrac{-34\,(-0.1985) - 4}{8}\right]$ by putting value of θc

From (6) $\theta a = \left[\dfrac{-6 \text{ x } 0.3435 - 2}{12}\right]$

$\boxed{\theta b = +0.3435}$ radians.

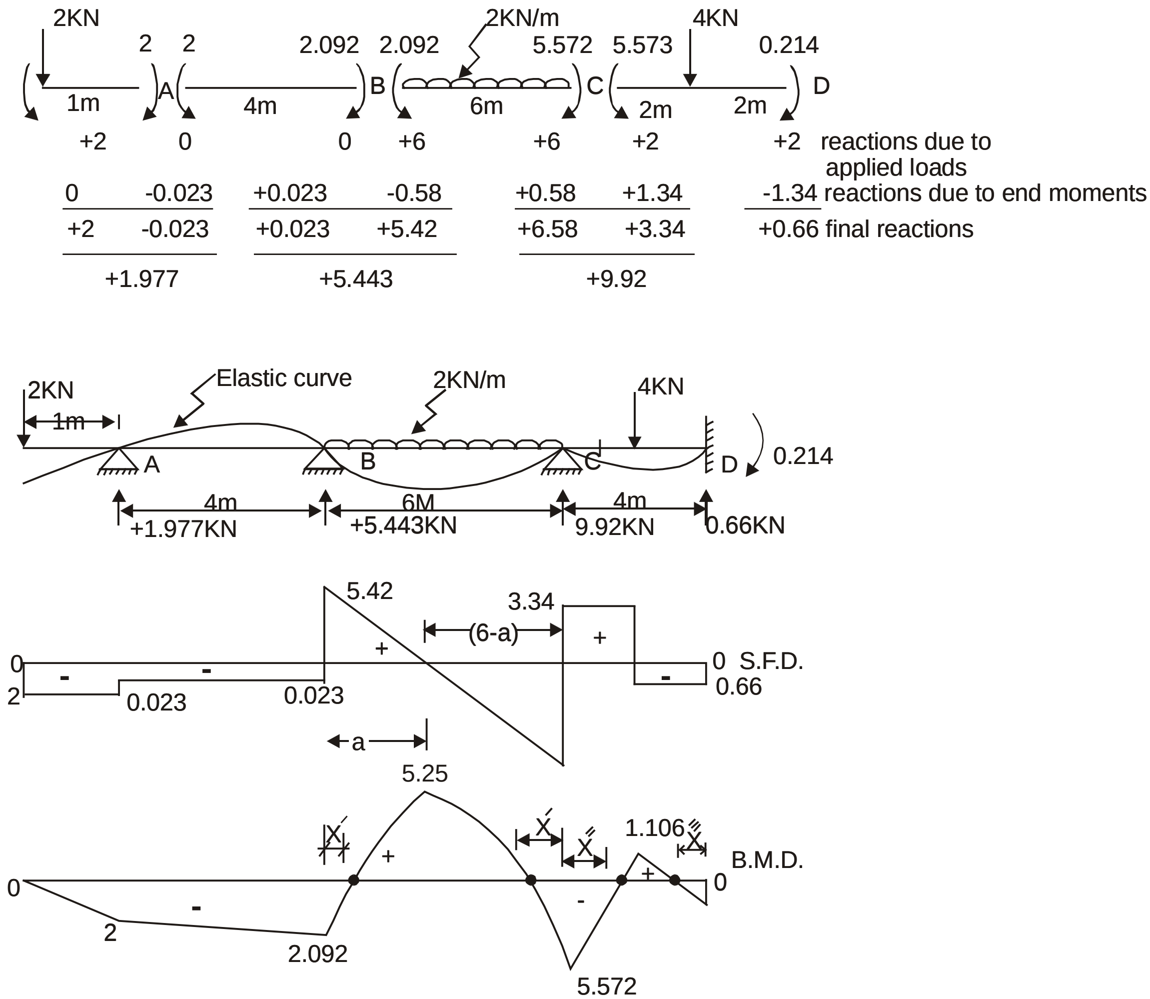
$\boxed{\theta a = -0.3384}$ radians.

Finally $\theta a = -0.3384$
$\theta b = +0.3435$
$\theta c = -0.1985$
$\theta d = 0$

Putting these values of rotations in simultaneous equations set up in step # (3) & getting the values of end moments as follows.

Mab = –12x (–0.3384) – 6 × 0..3435 = 1.9918 = + 2 KN–m
Mba = – 12x (+0.3435)– 6x(– 0.3384) = – 2.092 KN–m
Mbc = 6 – 16(+0.3435)–8 (– 0.1985) = + 2.092 KN–m
Mcb = – 6 – 16(– 0.1985) – 8(+0.3435) = – 5.572 KN–m
Mcd = 2 – 18 (– 0.1985) – 9 × 0 = + 5.573 KN–m
Mdc = – 2 – 18 x 0 – 9 (– 0.1985) = – 0.214 KN–m.

As the end moments have been calculated and they satisfy the joint conditions. Therefore, the structure is statically determinate at this stage. Reactions, S.F.D., B.M.D. & elastic curve can now be sketched.



TO LOCATE THE MAX. B.M. IN PORTION BC :–

$$\frac{5.42}{a} = \frac{6.58}{(6-a)}$$

$$5.42 \times 6 - 5.42.a = 6.58\,a$$
$$a = 2.71 \text{ m}$$
$$Mbc = -2.092 + \left(5.42 \times 2.71 - \frac{2}{2} \times 2.71^2\right) = +5.252 \text{ KN–m}$$
$$= 5.25 \text{ KN–m}$$

LOCATION OF POINTS OF CONTRAFLEXURE :–

Near B :– (Span BC)

$$-2.092 + 5.42\,X - X^2 = 0$$

$$X^2 - 5.42\,X + 2.092 = 0$$

$$X = \frac{5.42 \pm \sqrt{(5.42)^2 - 4 \times 1 \times 2.092}}{2}$$

$$X = \frac{5.42 \pm 4.583}{2}$$

$= 0.418$, 5.002 , So $\boxed{X = 0.418 \text{ m}}$

Near C :– Span BC

$$5.572 + 6.58\,X' - X'^2 = 0$$

$$X'^2 - 6.58\,X' + 5.572 = 0$$

$$X' = \frac{+\,6.58 \pm \sqrt{(6.58)^2 - 4 \times 1 \times 5.572}}{2}$$

$$= \frac{6.58 \pm 4.583}{2}$$

$X' = 0.998$, 5.582 $\boxed{X' = 0.998 \text{ m}}$

Near C : (Span CD)

$$5.573 + 3.34\,X'' = 0$$

$$\boxed{X'' = 1.669 \text{ m}}$$

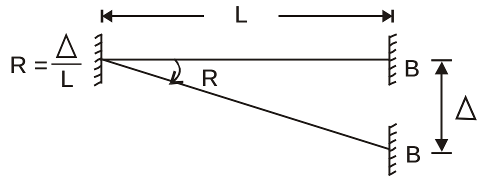
Near D :– (Span CD)

$$0.214 + 0.66\,X = 0$$

$$\boxed{X = 0.324 \text{ m}}$$

These have been shown on BMD.

4.4. ANALYSIS OF INDETERMINATE BEAMS DUE TO MEMBER AXIS ROTATION (SETTLEMENT OF SUPPORTS) :–



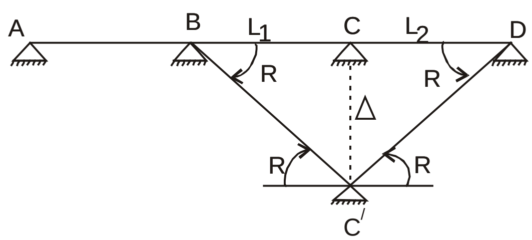
Consider a generalized fixed ended beam settling differentially at B. The angle R is measured from the original members axis to the displaced member axis and will be +ve if it is clockwise. The absolute values of $\frac{2EI}{L}$ with consistent units are to be used in the settlement problem and the final slope – deflection equation to be used for settlement problems is as follows:–

$$Mab = Mfab + \frac{2EI}{L}(-2\,\theta a - \theta b + 3\,R)$$

$$Mba = Mfba + \frac{2EI}{L}(-2\,\theta b - \theta a + 3\,R).$$

The above equation is general and can be used to find the end moments due to applied loading and due to sinking of supports simultaneously. However, it is a common practice to consider end moments induced due to applied loading separately from those induced due to settlement. The superposition principle can then be applied afterwards and the final end moments can be obtained.

If all supports of a continuous structure like beams and frames settle by the same amount, no additional end moments will be induced due to sinking. These will be induced only whenever there is a differential sinking of supports like the following case. Where support C sinks by Δ w.r.t supports B and D.



(Sign of R is the same if determined at the two ends of a span). So

Rab = 0 (Both supports of span AB are at the same level)

$Rbc = \frac{\Delta}{L_1}$ (Clock–wise angle is positive)

$Rcd = -\frac{\Delta}{L_2}$ (Counterclock–wise angle is negative)

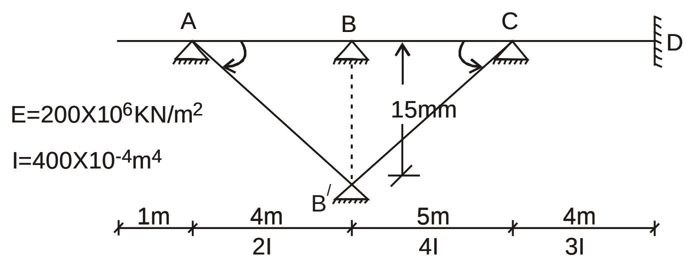
The following points are to be strictly followed :

(1) Consideration and computation of values of 'R' in the span effected by the settlement.

(2) Use proper sign for R keeping in view the corresponding sign convention.

(3) The units of the R.H.S. of the slope–deflection equation should be those of the B.M. (KN–m).

EXAMPLE NO. 4:– Analyze the continuous beam shown due to the settlement of support B by slope-deflection method. Draw shear and moment diagrams and sketch the elastic curve.



SOLUTION:–

Step 1: **Calculation of F.E.M :–**

$$\text{Mab} = \text{Mfab} + \frac{2EI}{L}(-2\,\theta a - \theta b + 3\,R). \text{ where R is in radians}$$

As there is no applied loading on the beam, therefore all fixed end moments terms in the slope – deflection equation will be equal to zero.

Step 2: **Calculation of R and $\frac{2EI}{L}$ terms for various spans :–**

Span AB.

$$R = +\frac{0.015}{4} = +3.75 \times 10^{-3} \text{ rad}$$

$$\frac{2EI}{L} = \frac{2x(200 \times 10^{6}) \times (2 \times 400 \times 10^{-6})}{4} \qquad \frac{KN/m^2 xm^4}{m}$$

= 80,000 KN–m

Span BC :–

$$R = -\frac{0.015}{5} = -3 \times 10^{-3} \text{ rad}$$

$$\frac{2EI}{L} = \frac{2\,(200 \times 10^{6})\,(4 \times 400 \times 10^{-6})}{5}$$

= 128,000 KN–m

Span CD :–

$R = 0$

$$\frac{2EI}{L} = \frac{2x\,(200 \times 10^{6}) \times (3 \times 400 \times 10^{-6})}{4}$$

$= 120,000$ KN–m

Step 3: **Write Slope–deflection Equation in terms of Joint Rotations & R.**

$Mab = 0 + 80,000\,(-2\,\theta a - \theta b + 11.25 \times 10^{-3})$

$Mba = 0 + 80,000\,(-2\,\theta b - \theta a + 11.25 \times 10^{-3})$

$Mbc = 128,000\,(-2\,\theta b - \theta c - 9 \times 10^{-3})$

$Mcb = 128,000\,(-2\,\theta c - \theta b - 9 \times 10^{-3})$

$Mcd = 120,000\,(-2\,\theta c - \theta d)$

$Mdc = 120,000\,(-2\theta d - \theta c)$

Step 4: **Joint Conditions (Conditions of Equilibrium + geometry) :–**

Joint A:–	$Mab = 0$	(Pin support)	$\rightarrow$ (1)
Joint B :–	$Mba+Mbc=0$	(Continuous support)	$\rightarrow$ (2)
Joint C :–	$Mcb + Mcd=0$	(Continuous support)	$\rightarrow$ (3)
Joint D :–	$\theta d = 0$	(Fixed support)	$\rightarrow$ (4)

Step 5: **Simultaneous Equations :–**

Putting joint conditions in slope – deflection equations

$-160,000\,\theta a - 80,000\,\theta b + 0 + 900 = 0$ $\quad \therefore$ Mab = 0 $\rightarrow$ (1)

$-160,000\,\theta b - 80,000\,\theta a + 900 - 256,000\,\theta b - 128,000\,\theta c - 1152 = 0$

$-80,000\,\theta a - 416,000\,\theta b - 128,000\,\theta c - 252 = 0$ $\quad \therefore$ Mba + Mbc = 0 $\rightarrow$ (2)

$-256,000\,\theta c - 128,000\,\theta b - 1152 - 240,000\,\theta c - 0 = 0$ $\quad$ Mcb + Mcd = 0 $\rightarrow$ (3)

$-128,000\,\theta b - 496,000\,\theta c - 1152 = 0$ $\quad$ Simplifying, finally

$-160,000\,\theta a - 80,000\,\theta b + 0 + 900 = 0$ $\rightarrow$ (1)

$-80,000\,\theta a - 416,000\,\theta b - 128,000\,\theta c - 252 = 0$ $\rightarrow$ (2)

$-128,000\theta b - 496,000\theta c - 1152 = 0$ $\rightarrow$ (3)

Solve the above three linear simultaneous equations to get the values of θa, θb & θc which will be put in the original slope–deflection equations to determine the final end moments.

From (1) $\qquad \theta a = \left(\frac{900 - 80000\,\theta b}{160000}\right)$

or $\qquad \theta a = 5.625 \times 10^{-3} - 0.5\,\theta b$ $\rightarrow$ (4)

From (3) $\theta c = \left(\dfrac{-128000\ \theta b - 1152}{496000}\right)$

so $\theta c = -0.258\ \theta b - 2.32 \times 10^{-3}$ → (5)

Put (4) and (5) in (2) , we have.

$-80,000\ [5.625 \times 10^{-3} - 0.5\ \theta b] - 416,000\ \theta b - 128,000$
$[-0.258\ \theta b - 2.32 \times 10^{-3}] - 252 = 0$

$-450 + 40,000\ \theta b - 416,000\ \theta b + 33,024\ \theta b + 296.96 - 252 = 0$

$$\theta b = \frac{-405.04}{342976}$$

$\theta b = -1.181 \times 10^{-3}$ radians.

Put θb in (1) because θa is dominant there.

$-160,000\ \theta a - 80,000\ (-1.181 \times 10^{-3}) + 900 = 0$

$$\theta a = \left[\frac{900 - 80000\ (-1.181 \times 10^{-3})}{160000}\right]$$

$\theta a = +6.215 \times 10^{-3}$ radians.

Put θb in (3) because θc is dominant there, we get.

$$\theta c = \frac{-128000\ (-1.181 \times 10^{-3}) - 1152}{496000}$$

$\theta c = -2.018 \times 10^{-3}$ red.

$\theta a = +6.215 \times 10^{-3}$ red.

$\theta b = -1.181 \times 10^{-3}$ red.

$\theta c = -2.018 \times 10^{-3}$ red.

$\theta d = 0$ red.

Step 6: **End Moments :–** Putting values of rotations in generalized slope – deflection equation.

$M_{ab} = 80,000\ (-2 \times 6.215 \times 10^{-3} + 1.181 \times 10^{-3} + 11.25 \times 10^{-3}) = 0$ KN–m (Check)

$M_{ba} = 80,000\ (+2 \times 1.181 \times 10^{-3} - 6.215 \times 10^{-3} + 11.25 \times 10^{-3}) = +592$ KN–m

$M_{bc} = 128,000\ (+2 \times 1.181 \times 10^{-3} + 2.018 \times 10^{-3} - 9 \times 10^{-3}) = -592$ KN–m

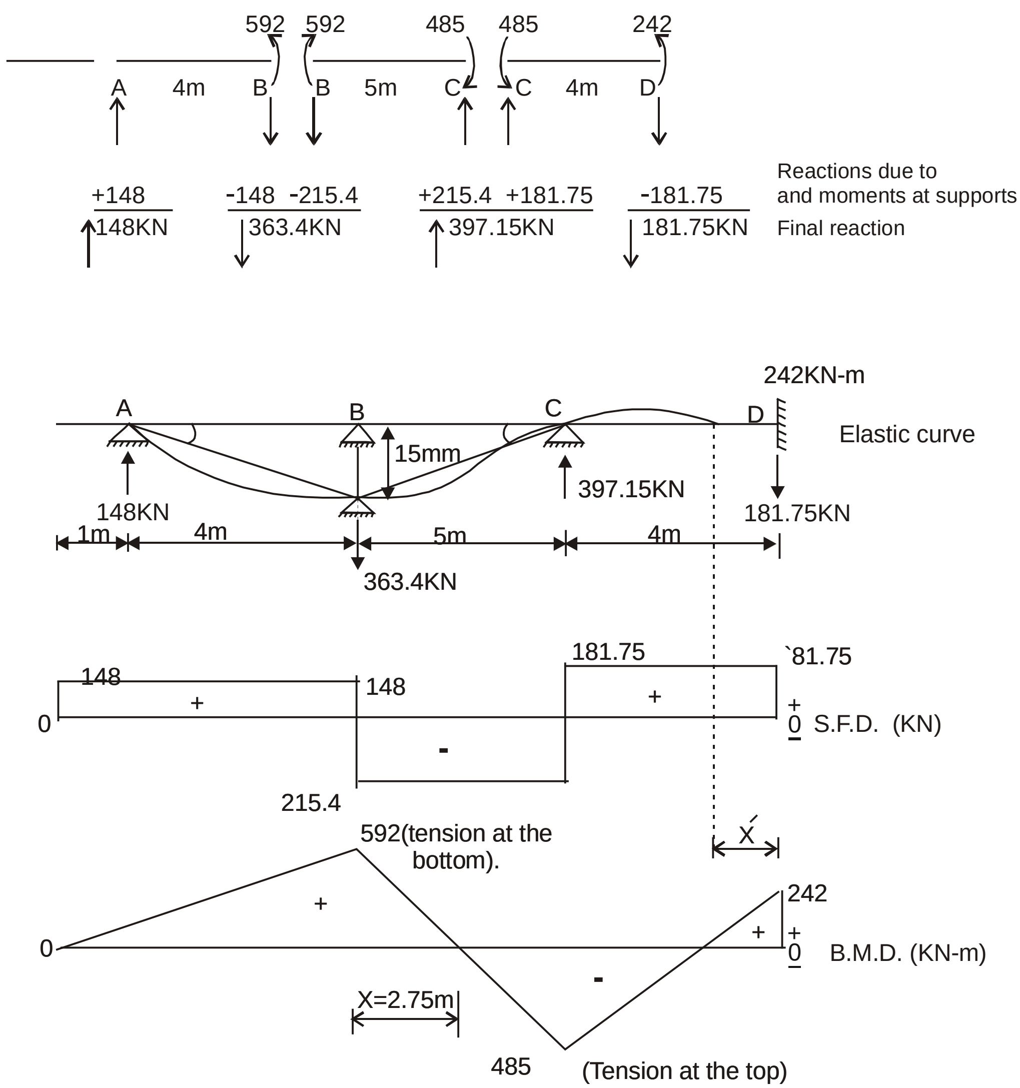
(**Note:** Mba = Mbc Check is OK)

$$Mcb = 128{,}000\,(+2 \times 2.018 \times 10^{-3} + 1.181 \times 10^{-3} - 9 \times 10^{-3}) = -\,485 \text{ KN–m}$$

$$Mcd = 120{,}000\,(+2 \times 2.018 \times 10^{-3} - 0\,) = +\,485 \text{ KN–m}$$

$$Mdc = 120{,}000\,(0+2.018 \times 10^{-3}\,) = +\,242 \text{ KN–m}$$

Note:- **A great care should be exercised while putting the direction of end moments in the free body diagrams and then drawing the composite B.M.D. e.g., a (+ve) end moment would mean that it is counterclockwise at that particular joint or vice versa. After putting the correct directions according to the sign convention, we will decide by the nature of B.M. strictly by keeping in view the sign convention for B.M. (tension at a bottom means +ve B.M.).**



POINTS OF CONTRAFLEXURES:–

Near B. Span BC

Let it be X.

$$MX = 592 - 215.4\,X = 0$$
$$X = 2.75\text{ m}$$

Near D. Span DC

Let it be X′

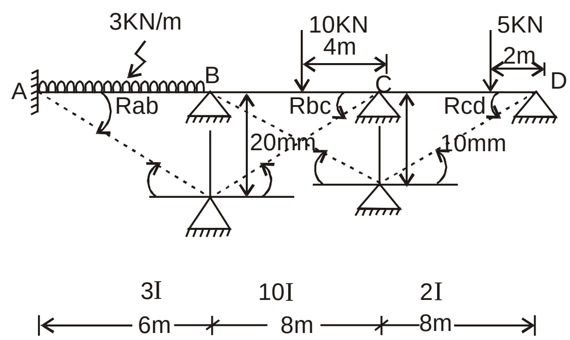
$$MX' = 242 - 181.75\,X' = 0$$
$$X' = 1.33\text{ m}$$

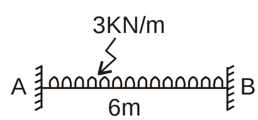
EXAMPLE NO. 5:- Analyze the following beam by slope – deflection method. Draw shear and moment diagrams. Sketch elastic curve.

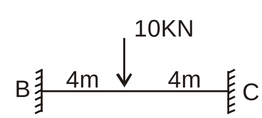
Take $I = 400 \times 10^{-6} m^4$

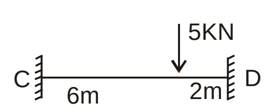
and $E = 200 \times 10^{6}$ KN/m^2.

SOLUTION :– Consider each span fixed end and compute fixed ended moments. This is a case of continuous beam carrying loads and subjected to settlements.









Step 1: **FIXED END MOMENTS**

$Mfab = 3 \times 6^2 / 12 = 9$ KN–m , $Mfba = -9$ KN–m

$Mfbc = 10 \times 4^2 \times 4 / 8^2 = 10$, $Mfcb = -10$ KN–m

$Mfcd = 5 \times 2^2 \times 6 / 8^2 = 1.875$, $Mfdc = -5 \times 6^2 \times 2/8^2 = -5.625$ KN–m

Step 2: **CALCULATION OF R & 2EI/L TERMS FOR VARIOUS SPANS :–**

SPAN AB :–

$$R = \frac{+0.020}{6} = +3.33 \times 10^{-3} \text{ rad.}$$

$$\frac{2EI}{L} = \frac{2 \times 200 \times 10^6 \times (3 \times 400 \times 10^{-6})}{6} = 80{,}000 \text{ KN–m}$$

SPAN BC :–

$$R = \frac{-0.02}{8} + \frac{0.01}{8} = -1.25 \times 10^{-3} \text{ rad}$$

$$\frac{2EI}{L} = \frac{2 \times 200 \times 10^6 \times (10 \times 400 \times 10^{-6})}{8} = 200{,}000 \text{ KN–m}$$

SPAN CD:–

$$R = \frac{-0.01}{8} = -1.25 \times 10^{-3} \text{ rad}$$

$$\frac{2EI}{L} = \frac{2 \times 200 \times 10^6 \times (2 \times 400 \times 10^{-6})}{8} = 40{,}000 \text{ KN–m}$$

Step 3: **SLOPE – DEFLECTION EQUATIONS:–**

Put values of fixed ended moments, Krel and R, we get.

$Mab = 9 + 80{,}000 (-2\theta a - \theta b + 10 \times 10^{-3})$.

$Mba = -9 + 80{,}000 (-2\theta b - \theta a + 10 \times 10^{-3})$

$Mbc = 10 + 200{,}000 (-2\theta b - \theta c - 3.75 \times 10^{-3})$.

$Mcb = -10 + 200{,}000 (-2\theta c - \theta b - 3.75 \times 10^{-3})$.

$Mcd = 1.875 + 40{,}000 (-2\theta c - \theta d - 3.75 \times 10^{-3})$.

$Mdc = -5.625 + 40{,}000 (-2\theta d - \theta c - 3.75 \times 10^{-3})$.

Step 4: **JOINT CONDITIONS :–**

Joint A $\Rightarrow$ $\theta a = 0$ (Fixed support)

Joint B $\Rightarrow$ $Mba + Mbc = 0$ (Continuous support)

Joint C $\Rightarrow$ $Mcb + Mcd = 0$ (Continuous support)

Joint D $\Rightarrow$ $Mdc = 0$ (Pin support)

Step 5: **SIMULTANEOUS EQUATIONS :–**

Putting values of Mba, Mbc, Mcb , Mcd and Mdc in terms of θ

$-9 - 160,000\,\theta b + 800 + 10 - 400,000\,\theta b - 200,000\,\theta c - 750 = 0$ Mba + Mbc = 0 and θa = 0

$-560,000\,\theta b - 200,000\,\theta c + 51 = 0$ → (1)

$-10 - 400,000\,\theta c - 200,000\,\theta b - 750 + 1.875 - 80,000\,\theta c - 40,000\,\theta d - 150 = 0$

$-200,000\,\theta b - 480,000\,\theta c - 40,000\,\theta d - 908.125 = 0$ Mcb + Mcd = 0 → (2)

$-5.625 - 80,000\,\theta d - 40,000\,\theta c - 150 = 0$ Mdc = 0

$-40,000\,\theta c - 80,000\,\theta d - 155.625 = 0$ → (3) Writing again

$-560,000\,\theta b - 200,000\,\theta c + 51 = 0$ → (1)

$-200,000\,\theta b - 480,000\,\theta c - 40,000\,\theta d - 908.125 = 0$ → (2)

$-40,000\,\theta c - 80,000\,\theta d - 155.625 = 0$ → (3)

From (1) $\theta b = \left(\dfrac{51 - 200000\,\theta c}{560000}\right)$ → (4)

From (3) $\theta d = \left(\dfrac{-155.625 - 40000\,\theta c}{80000}\right)$ → (5)

Put θb and θd in equ. (2) $-200,000\left(\dfrac{51 - 200000\,\theta c}{560000}\right) - 480,000\,\theta c$

$-40,000\left(\dfrac{-155.625 - 40000\,\theta c}{80000}\right) - 908.125 = 0$ Simplifying

$-18.2143 + 71428.5714\,\theta c - 480,000\,\theta c + 77.8125 + 200000\,\theta c - 908.125 = 0$

$-388571.4286\,\theta c - 848.5268 = 0$ we get θc = –21.8371 rad. From (4) and (5) θb and θd are calculated.

$\theta c = -21.8371 \times 10^{-4}$ rad.
$\theta b = +8.7097 \times 10^{-4}$ rad.
$\theta d = -8.5346 \times 10^{-4}$ rad.

Step 6: **END MOMENTS :–**

$Mab = 9 + 80,000\,(-8.7097 \times 10^{-4} + 10 \times 10^{-3}) = +739.32$ KN–m

$Mba = -9 + 80,000\,(-2 \times 8.7097 \times 10^{-4} + 10 \times 10^{-3}) = +651.64$ KN–m

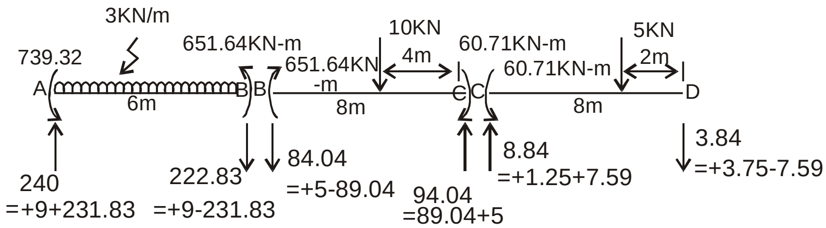
$Mbc = 10 + 200,000\,(-2 \times 8.7097 \times 10^{-4} + 21.8371 \times 10^{-4} - 3.75 \times 10^{-3}) = -651.64$ KN–m

$Mcb = -10 + 200,000\,(+2 \times 21.8381 \times 10^{-4} - 8.7097 \times 10^{-4} - 3.75 \times 10^{-3}) = -60.71$ KN–m

$Mcd = 1.875 + 40,000\,(+2 \times 21.8371 \times 10^{-4} + 8.5346 \times 10^{-4} - 3.75 \times 10^{-3}) = +60.71$ KN–m

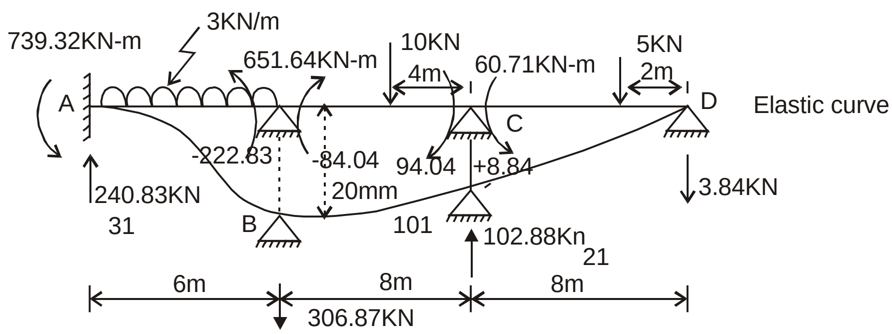
$Mdc = -5.625 + 40,000\,(+2 \times 8.5346 \times 10^{-4} + 21.8371 \times 10^{-4} - 3.75 \times 10^{-3}) = 0$ KN–m

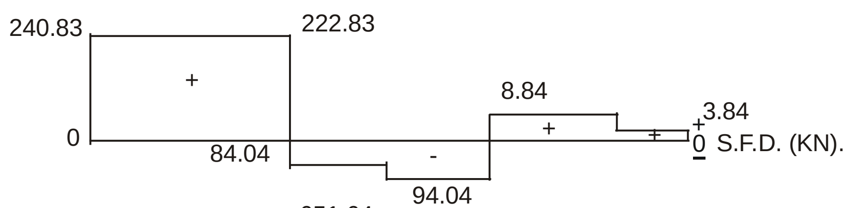
Step 7: **SUPPORT REACTIONS:–** By applying loads and end moments on free-body diagrams.

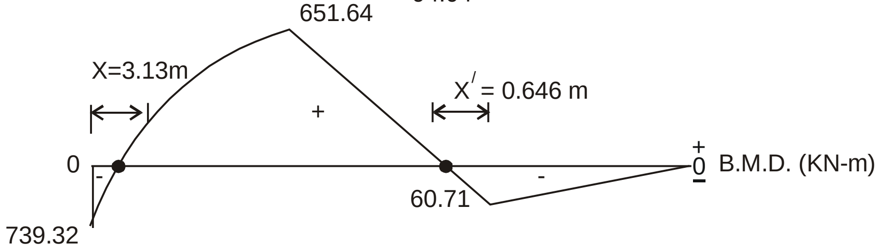


Net reactions, shear force and bending moment diagrams can now be plotted

Step 8: **S.F & B.M. DIAGRAMS & ELASTIC CURVE :–**







Step 9: **POINTS OF CONTRAFLEXURE :–**

NEAR A: Let it be at X from A in Span AB

$$MX = -739.32 + 240.83X - 1.5X^2 = 0$$

$$1.5X^2 - 240.83X + 739.32 = 0$$

$$X = \frac{+240.83 \pm \sqrt{(-240.83)^2 - 4 \times 1.5 \times 739.32}}{2 \times 1.5}$$

$$= \frac{240.83 \pm 231.44}{3}$$

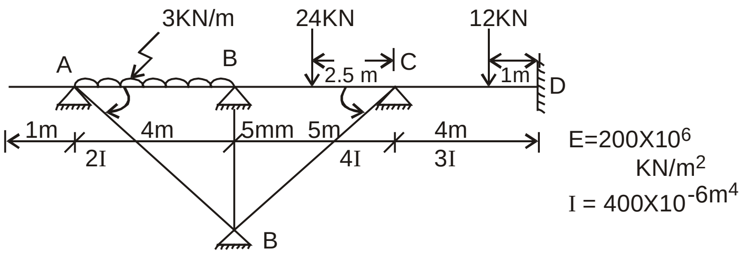
$$= 3.13 \quad , \quad 157.42$$

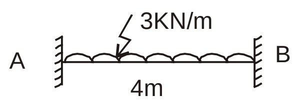
$$X = 3.13 \text{ m}$$

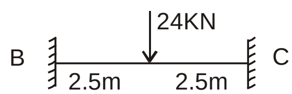
NEAR C: Let it be at X′ from C in Span BC – 60.71 + 94.04 X' = 0, X' = 0.646 m

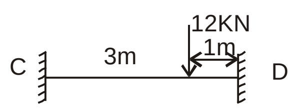
EXAMPLE NO.6:– Analyze the continuous beam shown due to settlement of support B by slope–deflection method. Draw S.F. & B.M. diagrams & sketch the elastic curve.

SOLUTION –









Step 1: **FIXED END MOMENTS**

$M_{fab} = 3 \times 4^2/12 = 4$ KN–m , $M_{fba} = -4$ KN–m

$M_{fbc} = 24 \times 2.5^2 \times 2.5/5^2 = 15$, $M_{fcb} = -15$ KN–m

$M_{fcd} = 12 \times 1^2 \times 3/4^2 = 2.25$, $M_{fdc} = -12 \times 3^2 \times 1/4^2 = -6.75$ KN–m

Step2: **CALCULATION OF R & $\frac{2EI}{L}$ TERMS FOR VARIOUS SPANS:–**

Span AB :–

$$R = +\frac{0.015}{4} = 3.75 \times 10^{-3} \text{ rad}$$

$$\frac{2EI}{L} = \frac{2(200 \times 10^6)(2 \times 400 \times 10^{-6})}{4} = 80{,}000 \text{ KN–m}$$

Span BC :–

$$R = -\frac{0.015}{5} = -3 \times 10^{-3} \text{ rad.}$$

$$\frac{2EI}{L} = \frac{2\,(200 \times 10^{6})\,(4 \times 400 \times 10^{-6})}{5}$$

$$= 128{,}000 \text{ KN–m}$$

Span CD :–

$$R = 0$$

$$\frac{2EI}{L} = \frac{2 \times (200 \times 10^{6})\,(3 \times 400 \times 10^{-6})}{4}$$

$$= 120{,}000 \text{ KN–m}$$

Step 3: **SLOPE – DEFLECTION EQUATIONS.**

Putting values of fixed end moments, $\frac{2EI}{L}$ and 3R we have.

$Mab = 4 + 80{,}000\,(-2\,\theta a - \theta b + 11.25 \times 10^{-3})$

$Mba = -4 + 80{,}000\,(-2\,\theta b - \theta a + 11.25 \times 10^{-3})$

$Mbc = 15 + 128{,}000\,(-2\,\theta b - \theta c - 9 \times 10^{-3})$

$Mcb = -15 + 128{,}000\,(-2\theta c - \theta b - 9 \times 10^{-3})$

$Mcd = 2.25 + 120{,}000\,(-2\theta c - \theta d)$

$Mdc = -6.75 + 120{,}000\,(-2\,\theta d - \theta c)$

Step 4: **JOINT CONDITIONS :–**

Joint A ;	Mab = 0	(Pin support)	→ (1)
Joint B ;	Mba + Mbc = 0	(Continuous support)	→ (2)
Joint C ;	Mcb + Mcd = 0	(Continuous support)	→ (3)
Joint D ;	θd = 0	(Fixed end)	

Step 5: **SIMULTANEOUS EQUATIONS :–**

$4 - 160{,}000\,\theta a - 80{,}000\,\theta b + 900 = 0$ ∴ Mab = 0

$-160{,}000\,\theta a - 80{,}000\,\theta b + 904 = 0$ → (1)

$-4 - 160{,}000\,\theta b - 80{,}000\,\theta a + 900 + 15 - 256{,}000\,\theta b - 128{,}000\,\theta c - 1152 = 0$
Mba + Mbc = 0

$-80{,}000\,\theta a - 416{,}000\,\theta b - 128{,}000\,\theta c - 241 = 0$ → (2)

$-15 - 256{,}000\,\theta c - 128{,}000\,\theta b - 1152 + 2.25 - 240{,}000\,\theta c - 120{,}000\,\theta d = 0$ → (3)
Mcb + Mcd = 0

$-128{,}000\,\theta b - 496{,}000\,\theta c - 120{,}000 \times 0 - 1164.75 = 0$

or $-128{,}000\,\theta b - 496{,}000\,\theta c - 1164.75 = 0$ Putting θd = 0 → (3)

Finally the equations become

$$-160{,}000\ \theta a - 80{,}000\ \theta b + 904 = 0 \qquad \rightarrow (1)$$

$$-80{,}000\ \theta a - 416{,}000\ \theta b - 128{,}000\ \theta c - 241 = 0 \qquad \rightarrow (2)$$

$$-128{,}000\ \theta b - 496{,}000\ \theta c - 1164.75 = 0 \qquad \rightarrow (3)$$

From (1) $$\theta a = \left(\frac{904 - 80000\ \theta b}{160000}\right) \qquad \rightarrow (4)$$

From (3) $$\theta c = \left(\frac{-1164.75 - 128000\ \theta b}{496000}\right) \qquad \rightarrow (5)$$

Put θa & θc from (4) and (5) in (2)

$$-80{,}000\left[\frac{904 - 80000\ \theta b}{160000}\right] - 416{,}000\ \theta b - 128{,}000$$

$$\left[\frac{-1164.75 - 128000\ \theta b}{496000}\right] - 241 = 0$$

$$-452 + 40{,}000\ \theta b - 416{,}000\ \theta b + 300.58 + 33032.26\theta b - 241 = 0$$

$$-342967.74\ \theta b - 392.42 = 0$$

$$\theta b = -1.144 \times 10^{-3} \quad \text{radians}$$

From (4) $$\theta a = \left(\frac{904 + 80000 \times 1.144 \times 10^{-3}}{160000}\right)$$

$$\theta a = +6.222 \times 10^{-3} \text{ rad.}$$

From (5) $$\theta c = \left(\frac{-1164.75 + 128000 \times 1.144 \times 10^{-3}}{496000}\right) = -2.053 \times 10^{-3} \text{ radians.}$$

$\theta c = -2.053 \times 10^{-3}$ rad.
$\theta a = +6.222 \times 10^{-3}$ rad.
$\theta b = -1.144 \times 10^{-3}$ rad.
$\theta c = -2.053 \times 10^{-3}$ rad.
$\theta d = 0$ rad.

Step 6: **END MOMENTS –**

Putting the values of Fixed end moments, relative stiffness, and end rotations (θ values) in slope-deflection equations, we have.

$Mab = 4 + 80{,}000\ (-2 \text{ x } 6.222 \times 10^{-3} + 1.144 \times 10^{-3} + 11.25 \times 10^{-3}) = 0$ KN-m

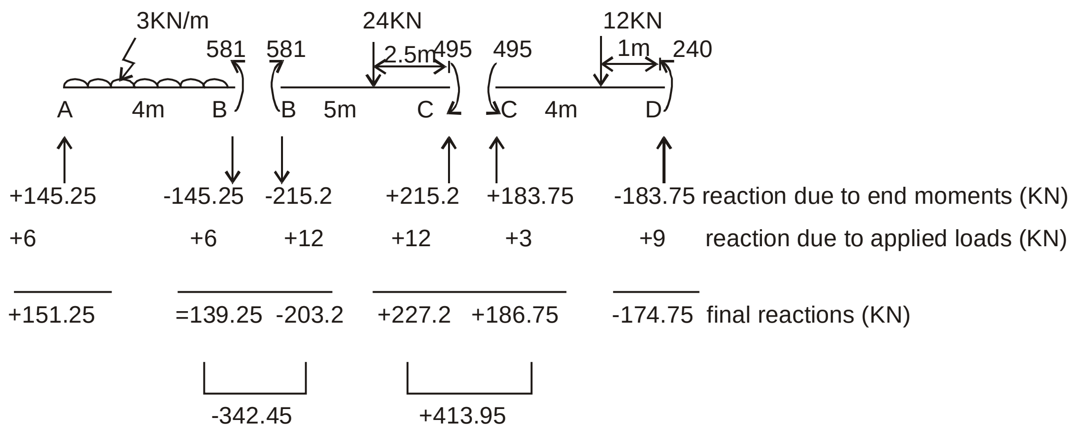
$Mba = -4 + 80{,}000\ (+2 \times 1.144 \times 10^{-3} - 6{,}222 \times 10^{-3} + 11.25 \times 10^{-3}) = +581$ KN-m

$Mbc = 15 + 128{,}000\ (+2 \times 1.144 \times 10^{-3} + 2.053 \times 10^{-3} - 9 \times 10^{-3}) = -581$ KN-m

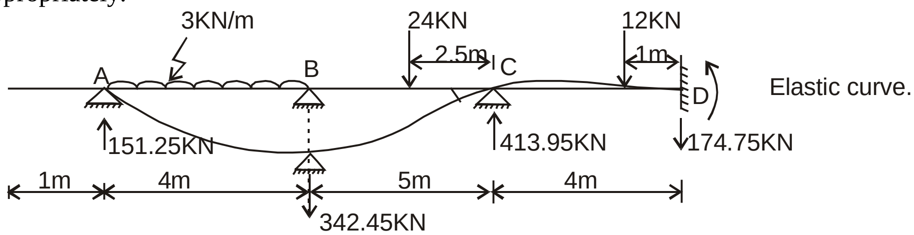
$Mcb = -15 + 128{,}000\ (+2 \times 2.053 \times 10^{-3} + 1.144 \times 10^{-3} - 9 \times 10^{-3}) = -495$ KN-m

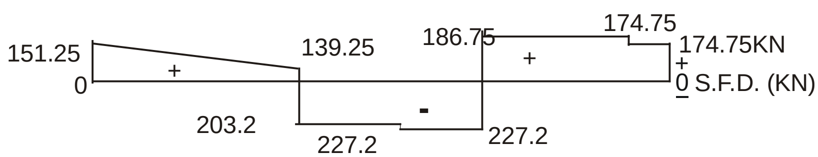
$$Mcd = 2.25 + 120{,}000\,(+\,2 \times 2.053 \times 10^{-3}) = +\,495 \text{ KN-m}$$
$$Mdc = -\,6.75 + 120{,}000\,(+2.053 \times 10^{-3}) \;=\; -495 \text{ KN-m}$$

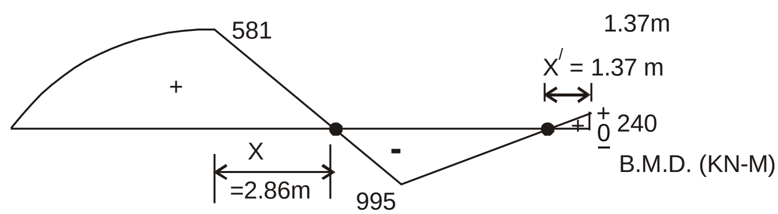
Now plot SFD, BMD and sketch elastic curve by applying loads and end moments to free-body diagram.



Note: Reactions due to loads and end moments have been calculated separately and then added up appropriately.







POINTS OF CONTRAFLEXURES :–

Near B :– Span AB

Let it be 'X'

$MX = 581 - 203.2\,X = 0$

$X = 2.86$ m

Near D :– Span CD

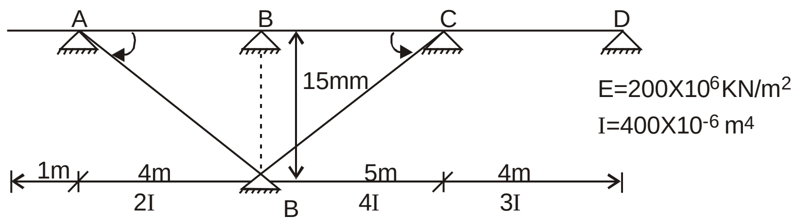
Let it be X′

Mx′ = 240 – 174.75 X′ = 0

X′ = 1.37 m These have been shown on BMD.

EXAMPLE NO. 7:– Analyze the continuous beam shown due to the settlement of support B alone by slope–deflection method. Draw S.F. & B.M. diagrams & sketch the elastic curve.

SOLUTION :–



Step 1: **FIXED END MOMENTS :–**

$$Mab = Mfab + \frac{2EI}{L}(-2\,\theta a - \theta b + 3R)$$ – A generalized slope–deflection equation.

As there is no applied loading on the beam, therefore, all fixed end moment terms in the slope–deflection equation will be equal to zero.

Step 2: **CALCULATION OF R AND $\frac{2EI}{L}$ TERMS FOR VARIOUS SPANS**

Span AB :–

$$R = +\frac{0.015}{4} = +3.75 \times 10^{-3} \text{ rad.}$$

$$\frac{2EI}{L} = \frac{2(200 \times 10^{6})(2 \times 400 \times 10^{-6})}{4} = 80{,}000 \text{ KN–m}$$

Span BC :–

$$R = -\frac{0.015}{5} = -3 \times 10^{-3} \text{ rad.}$$

$$\frac{2EI}{L} = \frac{2(200 \times 10^{6})(4 \times 400 \times 10^{-6})}{5} = 128{,}000 \text{ KN–m}$$

Span CD :–

R = 0 rad. (Both points C and D are at the same level)

$$\frac{2EI}{L} = \frac{2(200 \times 10^{6})(3 \times 400 \times 10^{-6})}{4} = 120{,}000 \text{ KN–m}$$

Step 3: **SLOPE–DEFLECTION EQUATIONS :–**

Putting $\frac{2EI}{L}$ and 3R values, we have.

$$Mab = 80,000\ (-2\theta a - \theta b - 11.25 \times 10^{-3})$$

$$Mba = 80,000\ (-2\ \theta b - \theta a + 11.25 \times 10^{-3})$$

$$Mbc = 128,000\ (-2\ \theta b - \theta c - 9 \times 10^{-3})$$

$$Mcb = 128,000\ (-2\ \theta c - \theta b - 9 \times 10^{-3})$$

$$Mcd = 120,000\ (-2\ \theta c - \theta d)$$

$$Mdc = 120,000\ (-2\ \theta d - \theta c)$$

Step 4: **JOINT CONDITIONS :–**

Joint A ; Mab = 0	(Pin support)
Joint B ; Mba + Mbc = 0	(Continuous support)
Joint C ; Mcb + Mcd = 0	(Continuous support)
Joint D ; Mdc = 0	(Pin support)

Step 5: **SIMULTANEOUS EQUATIONS :–**

Putting joint conditions in Slope – deflection equation, we have

(Mab = 0)

$$-160,000\ \theta a - 80,000\ \theta b + 900 = 0 \qquad \rightarrow (1)$$

(Mba + Mbc = 0)

$$-160,000\ \theta b - 80,000\ \theta a + 900 - 256,000\ \theta b - 128,000\ \theta c - 1152 = 0$$

$$-80,000\ \theta a - 416,000\ \theta b - 128,000\ \theta c - 252 = 0 \qquad \rightarrow (2)$$

(Mcb + Mcd = 0)

$$-256,000\ \theta c - 128,000\ \theta b - 1152 - 240,000\ \theta c - 120,000\ \theta d = 0$$

$$-128,000\ \theta b - 496,000\ \theta c - 120,000\ \theta d - 1152 = 0 \qquad \rightarrow (3)$$

(Mdc = 0)

$$-240,000\ \theta d - 120,000\ \theta c = 0$$

$$-120,000\ \theta c - 240,000\ \theta d = 0 \qquad \rightarrow (4)$$

Re-writing

$$-160,000\ \theta a - 80,000\ \theta b + 0 + 0 + 900 = 0 \qquad \rightarrow (1)$$

$$-80,000\ \theta a - 416.000\ \theta b - 128,000\ \theta c + 0 - 252 = 0 \qquad \rightarrow (2)$$

$$0 \quad - 128,000\ \theta b - 496,000\ \theta c - 120,000\ \theta d - 1152 = 0 \qquad \rightarrow (3)$$

$$0 \quad + \quad 0 \quad - 120,000\ \theta c - 240,000\ \theta d + 0 = 0 \qquad \rightarrow (4)$$

From (1) $\qquad \theta a = \left(\frac{900 - 80000\ \theta b}{160000}\right) \qquad \rightarrow (5)$

From (4) $\qquad \theta d = \frac{-120000\ \theta c}{240000}$

$$\theta d = -0.5\ \theta c \qquad \rightarrow (6)$$

Put (5) in (2)

$$-80{,}000\left(\frac{900-80000\,\theta b}{160000}\right)-416{,}000\,\theta b-128{,}000\,\theta c-252=0$$

$$-50+40{,}000\,\theta b-416{,}000\,\theta b-128{,}000\,\theta c-252=0$$

$$-376{,}000\,\theta b-128{,}000\,\theta c-702=0 \qquad \rightarrow (7)$$

Put (6) in (3)

$$-128{,}000\,\theta b-496{,}000\,\theta c-120{,}000\,(-0.5\,\theta c)-1152=0$$

$$-128{,}000\,\theta b-436{,}000\,\theta c-1152=0 \qquad \rightarrow (8)$$

From (7)

$$\theta b=\left(\frac{-702-128000\,\theta c}{376000}\right) \qquad \rightarrow (9)$$

Put θb from equation (9) in (8), we have.

$$-128{,}000\left(\frac{-702-128000\,\theta c}{376000}\right)-436{,}000\,\theta c-1152=0$$

$$238.98+43574.47\,\theta c-436{,}000\,\theta c-1152=0$$

$$-392{,}425.53\,\theta c-913.02=0$$

$$\theta c=-2.327\times10^{-3} \quad \text{radians.}$$

from (9) $\theta b=\left(\frac{-702+128000\times2.327\times10^{-3}}{376000}\right)$

$\theta b=-1.075\times10^{-3}$ rad. Now calculate other rotations from equations.

from (5) $\theta a=\left(\frac{900+80000\times1.075\times10^{-3}}{160000}\right)$

$\theta a=+6.162\times10^{-3}$ rad.

from (6)

$\theta d=-0.5\,(-2.327\times10^{-3})$

$\theta d=+1.164\times10^{-3}$ rad.

Final values of end rotations are:

$\theta a=+6.162\times10^{-3}$ rad.

$\theta b=-1.075\times10^{-3}$ rad.

$\theta c=-2.327\times10^{-3}$ rad.

$\theta d=+1.164\times10^{-3}$ rad.

Step 6: **END MOMENTS :–**

Putting values of rotations in slope-deflection equations.

$M_{ab} = 80{,}000\,(-2 \times 6.162 \times 10^{-3} + 1.075 \times 10^{-3} + 11.25 \times 10^{-3} = 0$ KN-m

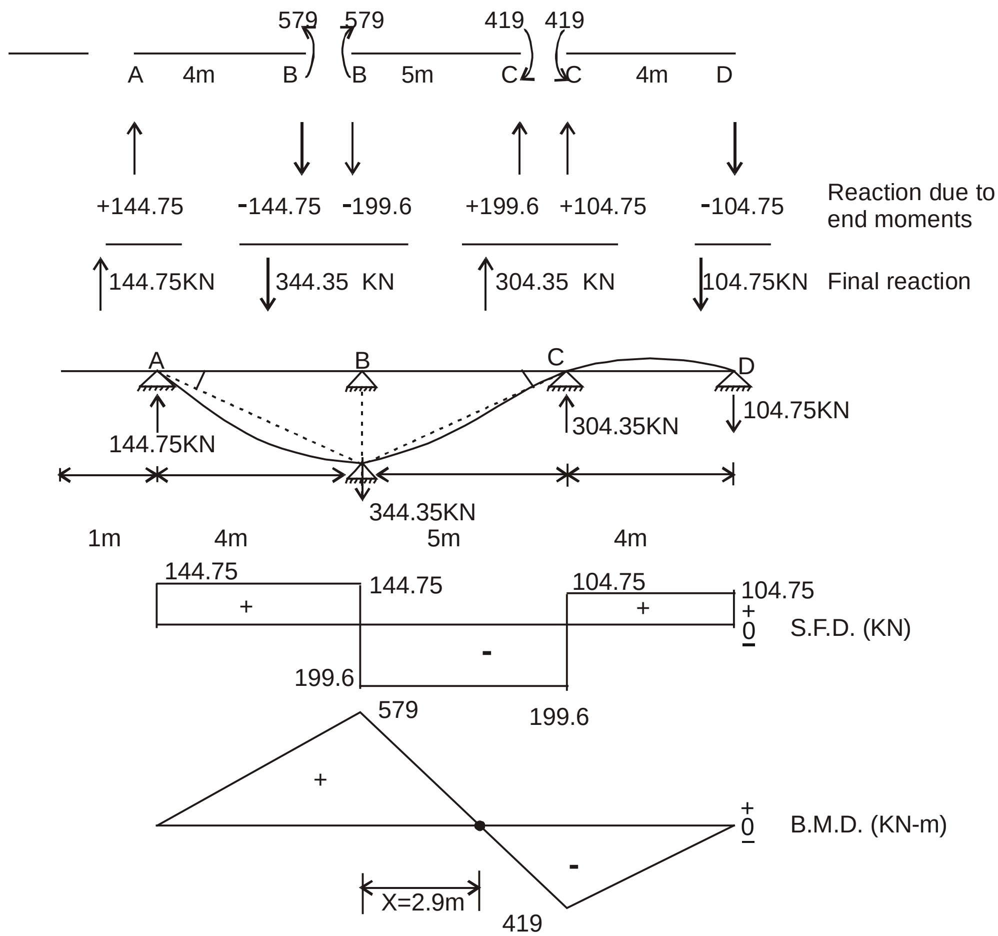
$M_{ba} = 80{,}000(+2 \times 1.075 \times 10^{-3} - 6.162 \times 10^{-3} + 11.25 \times 10^{-3}) = +579$ KN-m

$M_{bc} = 128{,}000\,(+2 \times 1.075 \times 10^{-3} + 2.327 \times 10^{-3} - 9 \times 10^{-3}) = -579$ KN-m

$M_{cb} = 128{,}000\,(+2 \times 2.327 \times 10^{-3} + 1.075 \times 10^{-3} - 9 \times 10^{-3}) = -419$ KN-m

$M_{cd} = 120{,}000\,(+2 \times 2.327 \times 10^{-3} - 1.164 \times 10^{-3}) = +419$ KN-m

$M_{dc} = 120{,}000\,(-2 \times 1.164 \times 10^{-3} + 2.327 \times 10^{-3}) = 0$ KN-m



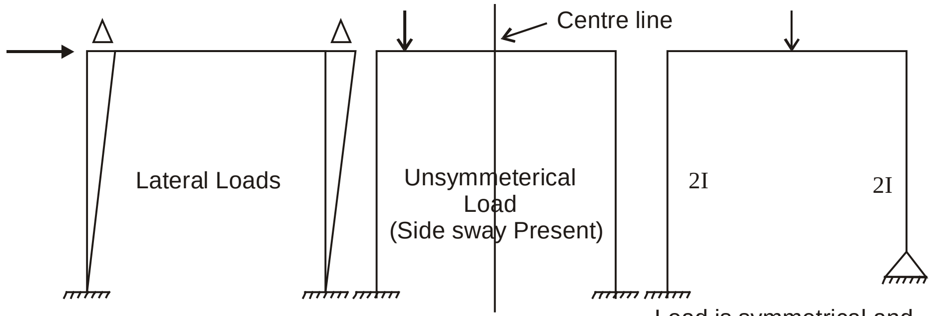
Near B :– Span BC

Let it be at 'X' from B.

$M_X = 579 - 199.6\,X = 0$

$X = 2.9$ m

4.5. APPLICATION TO FRAMES (WITHOUT SIDE SWAY):–



The side sway (relative displacement of two ends of a column) or the horizontal movement of the structure may become obvious once the structure and the loading is inspected in terms of inertia, E values and support conditions etc. However, following are the rules and guide lines which may be followed for deciding whether side sway is present or not.

(1) In case of symmetrical frames subjected to symmetrical loading, the side sway may be neglected for columns having equal inertia values if support conditions are same.

(2) If a force is applied in horizontal direction to a symmetrical frame where no arrangement exists for preventing horizontal movement, the side sway must be considered.(with reference to all these diagrams).

(3) An unsymmetrical frame subjected to symmetrical loading might be considered to have side sway.

4.6. UNSYMMETRICAL FRAME :–

"An unsymmetrical frame is that which has columns of unequal lengths and different end conditions and moment of inertia the load may be symmetrical or unsymmetrical."

4.7. STIFFNESS :–

"Stiffness can be defined as the resistance towards deformation which is a material, sectional and support parameter." More is the stiffness, less is the deformation & vice versa. Stiffness attracts loads / stresses.

The stiffness is of various types :

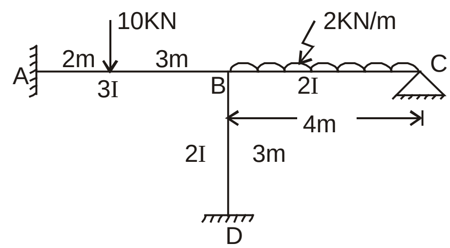
(1) Axial stiffness (AE).

(2) Flexural stiffness (EI).

(3) Shear stiffness (AG).

(4) Torsional stiffness (GJ).

EXAMPLE NO. 8:– Analyze the rigid frame shown by slope–deflection method.

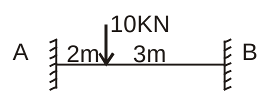


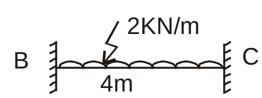
SOLUTION :– Examining loads and support conditions, horizontal moment is not possible.

Step 1: **Relative Stiffness :–**

Member	I	L	$\frac{I}{L}$	Krel.
AB	3	5	$\frac{3}{5}\times 30$	18
BC	2	4	$\frac{2}{4}\times 30$	15
BD	2	3	$\frac{2}{3}\times 30$	20

Step 2: **Fixed End Moments :–**





Mfab = $10 \times 3^2 \times 2/5^2$ = 7.2 KN–m , Mfba = $-\ 10 \times 2^2 \times 3/5^2$ = –4.8 KN-m

Mfbc = $2 \times 4^2/12$ = 2.67 KN-m, Mfcb = – 2.67 KN-m

Mfdb = Mfdb = 0 (There is no load acting within member BD)

Step 3: **Generalized Slope – deflection Equation :–**

Put values of fixed end moments.

$$M_{ab} = 7.2 + 18(-2\,\theta a - \theta b) = 7.2 - 36\,\theta a - 18\,\theta b$$

$$M_{ba} = -\,4.8 + 18\,(-\,2\,\theta b - \theta a) = -\,4.8 - 36\,\theta b - 18\,\theta a.$$

$$M_{bc} = 2.67 + 15\,(-\,2\,\theta b - \theta c) = 2.67 - 30\,\theta b - 15\,\theta c.$$

$$M_{cb} = -\,2.67 + 15\,(-2\,\theta c - \theta b) = -\,2.67 - 30\,\theta c - 15\,\theta b$$

$$M_{bd} = 0 + 20\,(-\,2\,\theta b - \theta d) = -\,40\,\theta b - 20\,\theta d$$

$$M_{db} = 0 + 20\,(-\,2\,\theta d - \theta b) = -\,40\,\theta d - 20\,\theta b.$$

Step 4: **Joint Conditions:–**

Joint A : $\theta a = 0$ (Being fixed end)

Joint B : Mba + Mbc + Mbd = 0 → (1) Continuous joint

Joint C : Mcb = 0 (Pin end) → (2)

Joint D : $\theta d = 0$ (Fixed end)

Step 5: **Simultaneous equations**

Putting above joint conditions in slope deflection equations, we have.

$$-4.8-36\,\theta b-18\,\theta a+2.67\;30\,\theta b-15\,\theta c-40\theta b-20\,\theta d=0 \rightarrow (1)$$

Mba + Mbc + Mbd = 0

Put $\theta d = 0$ and $\theta a = 0$.

$$-4.8-36\,\theta b-0+2.67-30\,\theta b-15\,\theta c-40\theta b-0=0$$

$$-106\,\theta b-15\,\theta c-2.13=0 \rightarrow (1)$$

(Mcb = 0)

$$-2.67-30\,\theta c-15\,\theta b=0 \rightarrow (2)$$

$$-15\,\theta b-30\,\theta c-2.67=0 \rightarrow (2)$$

$$-106\,\theta b-15\,\theta c-2.13=0 \rightarrow (1)$$

$$-15\,\theta b-30\,\theta c-2.67 \rightarrow (2)$$

Multiply (1) by 2 and subtract (2) from (1)

$$-212\,\theta b-30\,\theta c-4..26=0$$

$$\mp\;15\,\theta b\mp 30\,\theta c\mp 2.67=0$$

$$-197\,\theta b-1.59=0$$

$$\theta b=-8.07\times 10^{-3}\text{ rad.}$$

From (1) $\Rightarrow$ $-106\,(-8.07\times 10^{-3})-15\,\theta c-2.13=0$

$$\theta c=-84.96\times 10^{-3}\text{ rad.}$$

$$\theta a=0\text{ rad.}$$

$$\theta b=-8.07\times 10^{-3}\text{ rad.}$$

$$\theta c=-84.96\times 10^{-3}\text{ rad.}$$

$$\theta d=0\text{ rad.}$$

Step 6: **End moments.**

Putting values of FEM and rotations in slope-deflection equations.

$$Mab=7.2-36\,(0)-18(-8.07\times 10^{-3})=+7.345\text{ KN–m}$$

$$Mba=-4.8-36(-8.07\times 10^{-3})-18\,(0)=-4.509\text{ KN–m}$$

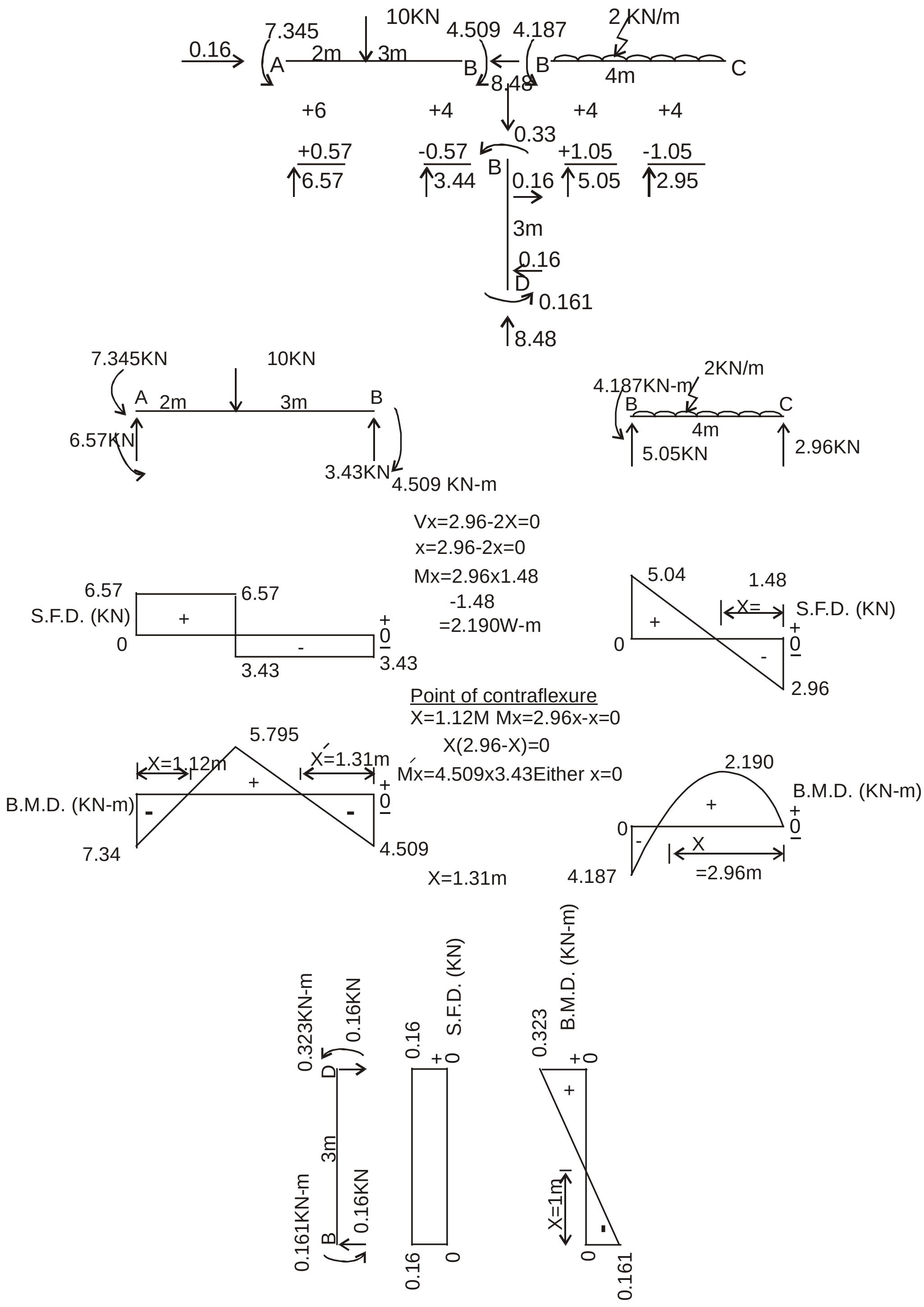
$$Mbc=2.67-30(-8.07\times 10^{-3})-15\,(-84.96\times 10^{-3})=+4.187\text{ KN–m}$$

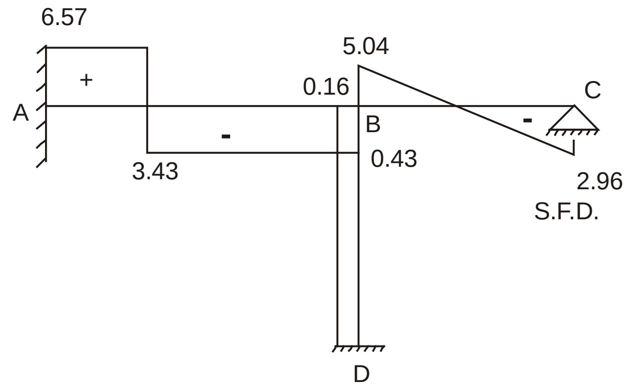
$$Mcb=-2.67-30\,(-84.96\times 10^{-3})-15\,(-8.07\times 10^{-3})=0$$

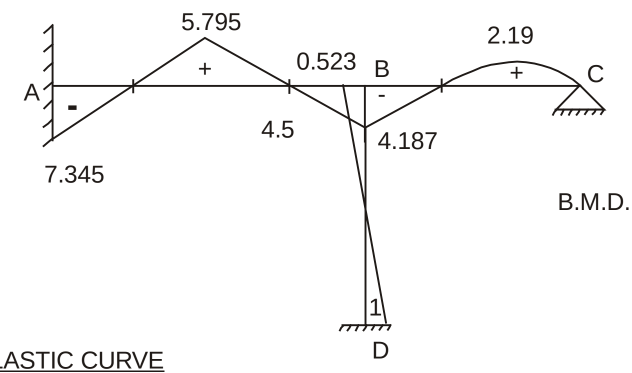
$$Mbd=-40(-8.07\times 10^{-3})-20\,(0)=+0.323\text{ KN–m}$$

$$Mdb=-40(0)-20\,(-8.07\times 10^{-3})=+0.161\text{ KN–m}$$

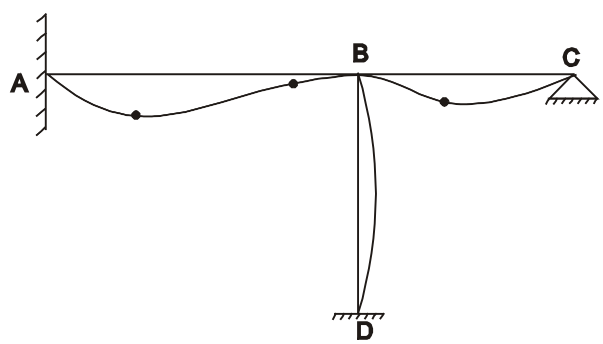
Draw SFD , BMD and sketch elastic curve.



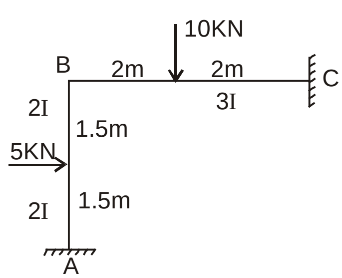




ELASTIC CURVE



EXAMPLE NO. 9:– Analyze the rigid frame shown by slope–deflection method



SOLUTION:– Inspecting loads and support conditions, horizontal displacement is not possible.

Step 1: **Relative Stiffness :–**

Member	I	L	$\frac{I}{L}$	Krel.
AB	2	3	$\frac{2}{3} \times 12$	8
BC	3	4	$\frac{3}{4} \times 12$	9

Step 2: **Fixed End Moments :–**

$$\text{Mfab} = \frac{5 \times 1.5^2 \times 1.5}{3^2} = +\,1.875 \text{ KN–m}$$

Mfba = – 1.875 KN–m

$$\text{Mfbc} = \frac{10 \times 2^2 \times 2}{4^2} = +\,5 \text{ KN–m}$$

Mfcb = – 5 KN–m

Step 3: **Generalized Slope–deflection Equations :–**

Put values of fixed end moments and Krel.

Mab = 1.875 + 8 (– 2 θa – θb)

Mba = – 1.875 + 8 (– 2 θb – θa)

Mbc = 5 + 9 (– 2 θb – θc)

Mcb = – 5 + 9 (– 2 θc – θb)

Step 4: **Joint Conditions :–**

Joint A : θa = 0 Being fixed End.

Joint B : Mba + Mbc = 0 Continuous end.

Joint C : θc = 0 Being fixed End.

Step 5: **Simultaneous Equations :–**

Put θa = θc = 0 in the joint condition at B.

Mba + Mbc = 0

– 1.875 – 16 θb – 0 + 5 – 18 θb – 0 = 0

3.125 – 34 θb = 0

θb = + 0.092 radians.

θa = 0

θc = 0

Step 6: **End moments.**

Put values of rotations in slope-deflection equations.

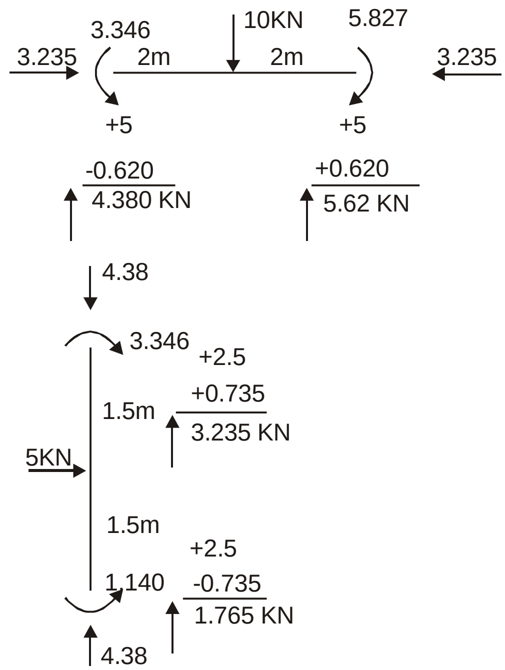
Mab = 1.875 + 8 (0 –0.092) = + 1.140 KN–m

Mba = – 1.875 + 8 (– 2 x 0.092 – 0) = – 3.346 KN–m

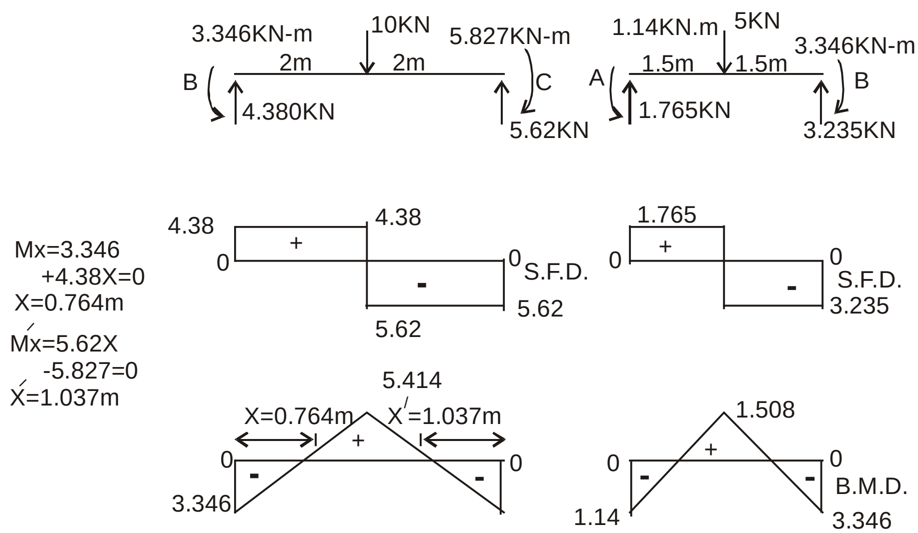
Mbc = 5 + 9 (– 2 x 0.092 – 0) = + 3.346 KN–m

Mcb = – 5 + 9 (0 – 0.092) = – 5.827 KN–m

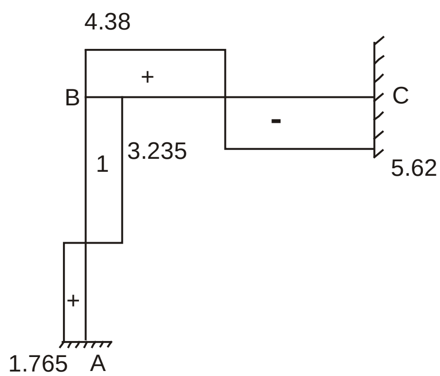
Now draw SFD , BMD and sketch elastic curve. Doing it by-parts for each member.



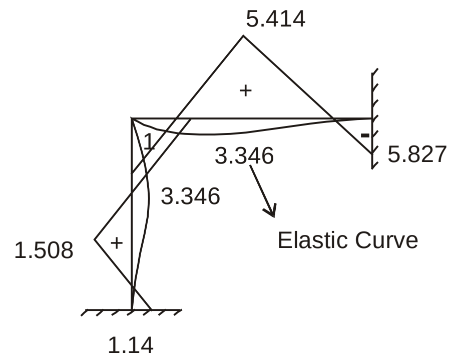
SHEAR FORCE AND B.M. DIAGRAMS



S.F.D



BMD

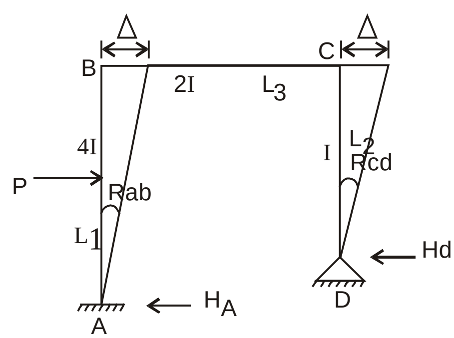


4.8. FRAMES WITH SIDE SWAY – SINGLE STOREY FRAMES :–

For columns of unequal heights, R would be calculated as follows:

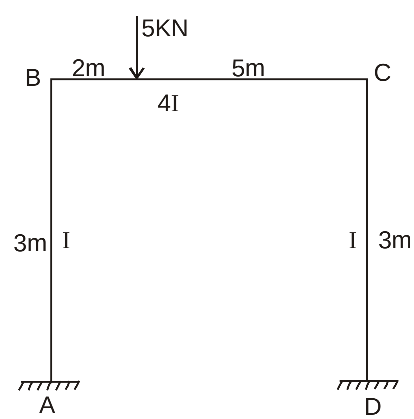
$$Rab = \frac{\Delta}{L_1}$$

$$Rcd = \frac{\Delta}{L_2}$$



To show the application to frames with sidesway, let us solve examples.

EXAMPLE NO. 10:– Analyze the rigid frame shown by slope–deflection method.



SOLUTION:–

Step 1: **Relative Stiffness :–**

Member	I	L	$\frac{I}{L}$	Krel.
AB	1	3	$\frac{1}{3} \times 21$	7
BC	4	7	$\frac{4}{7} \times 21$	12
CD	1	3	$\frac{1}{3} \times 21$	7

Step 2: **Relative Values of R :–**

$Rab = Rcd = \frac{\Delta}{3} = Rrel$ or R (columns are of 3m length)

$Mab = Mfab + Krel_{ab} (-2\,\theta a - \theta b + Rrel)$

$Mba = Mfba + Krel_{ab} (-2\,\theta b - \theta a + Rrel)$

Other expressions can be written on similar lines.

NOTE :– In case of side sway, R values are obtained for columns only because the columns are supposed to prevent (resist) side sway not beams.

Step 3: **Fixed End Moments :–**

$$Mfbc = \frac{5 \times 5^2 \times 2}{7^2} = 5.10 \text{ KN–m}$$

$$Mfcb = \frac{-5 \times 2^2 \times 5}{7^2} = -2.04 \text{ KN–m}$$

All other F.E.M. are zero because there are no loads on other Spans.

i.e. $Mfab = Mfba = 0$
& $Mfcd = Mfdc = 0$

Step 4: **Slope – deflection Equations :–** Putting values of FEM's while R will now appear as unknown.

$Mab = 0 + 7 (-2\theta a - \theta b + R)$

$Mba = 0 + 7 (-2\,\theta b - \theta a + R)$

$Mbc = 5.1 + 12 (-2\,\theta b - \theta c)$

$Mcb = -2.04 + 12 (-2\,\theta c - \theta b)$

$Mcd = 0 + 7 (-2\,\theta c - \theta d + R)$

$Mdc = 0 + 7 (-2\,\theta d - \theta c + R)$

Step 5: **Joint Conditions :–**

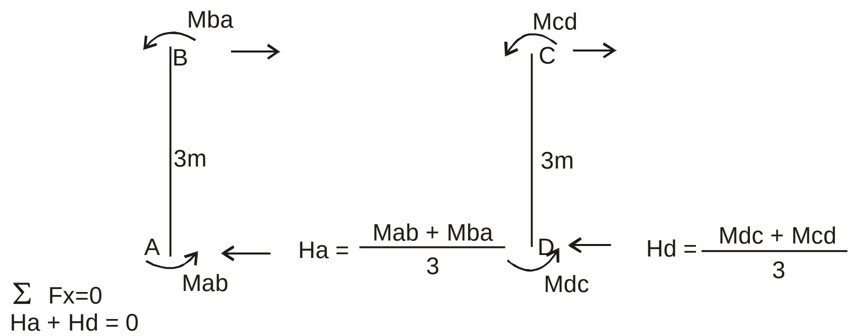
Joint A : $\theta a = 0$ (Fixed joint)

Joint B : Mba + Mbc = 0 (Continuous joint) → (1)

Joint C : Mcb + Mcd = 0 (Continuous joint) → (2)

Joint D : $\theta d = 0$ (Fixed joint)

Step 6: **Shear Conditions :–**



NOTE: Shear forces are in agreement with direction of Δ. The couple constituted by shears is balanced by the direction of end moments. (Reactive horizontal forces constitute a couple in opposite direction to that of end momens).

$\sum Fx = 0$

Ha + Hd = 0

Write in terms of moments.

Mab + Mba + Mdc + Mcd = 0 → (3)

Apply equations (1), (2) & (3) and solve for θb, θc & R. Equation (3) is also called shear condition.

Step 7: **Simultaneous Equations :–**

Put θa and θd equal to zero in joint conditions for B and C in terms of end moments.

Mba + Mbc = 0

so $-14\,\theta b + 7R + 5.1 - 24\,\theta b - 12\,\theta c = 0$ → (1)

Mcb + Mcd = 0

$-38\,\theta b - 12\,\theta c + 7R + 5.1 = 0$

$-2.04 - 24\,\theta c - 12\,\theta b - 14\,\theta c + 7R = 0$

or $-12\,\theta b - 38\,\theta c + 7R - 2.04 = 0$ → (2)

Mab + Mba + Mdc + Mcd = 0

$-7\,\theta b + 7R - 14\,\theta b + 7R - 7\,\theta c + 7R - 14\,\theta c + 7R = 0$

$-21\,\theta b - 21\,\theta c + 28R = 0$

or $-3\,\theta b - 3\,\theta c + 4R = 0$ → (3)

re-writing the equations again.

$-38\,\theta b - 12\,\theta c + 7R + 5.1 = 0$ → (1)

$-12\,\theta b - 38\,\theta c + 7R - 2.04 = 0$ → (2)

$-3\,\theta b - 3\,\theta c + 4R = 0$ → (3)

Subtract (2) from (1)

$-38\,\theta b - 12\,\theta c + 7\,R + 5.1 = 0$

$-12\,\theta b - 38\,\theta c + 7\,R - 2.04 = 0$

$-26\,\theta b + 26\,\theta c + 7.14 = 0 \quad \rightarrow (4)$

Multiply (2) by 4 & (3) by 7 & subtract (3) from (2)

$-48\,\theta b - 152\,\theta c + 28\,R - 8.16 = 0 \quad \rightarrow (2)$

$\mp 21\,\theta b \mp 21\,\theta c \pm 28\,R = 0 \quad \rightarrow (3)$

$-27\,\theta b - 131\,\theta c - 8.16 = 0 \quad \rightarrow (5)$

From (4)

$\theta b = \dfrac{26\,\theta c + 7.14}{26}$ put in (5) and solve for θc

$-27\,\dfrac{26\,\theta c + 7.14}{26} - 131\,\theta c - 8.16 = 0 \quad \rightarrow (6)$

$-27\,\theta c - 7.415 - 131\,\theta c - 8.16 = 0$

$-158\,\theta c - 15.575 = 0$

$\theta c = -0.0986$ rad.

From (6), $\theta b = \dfrac{-26 \times 0.0986 + 7.14}{26}$

$\theta b = +0.1760$ rad.

From (1)

$-38\,(0.1760) - 12\,(-0.0986) + 7R + 5.1 = 0$

$R = +0.0580$

So finally, we have.

$\therefore\ \theta a = 0$

$\theta b = +0.1760$

$\theta c = -0.0986$

$\theta d = 0$

$R = +0.0580$

END MOMENTS :–

Putting above values of rotations and R in slope deflection equations, we have.

Mab = 7 (0– 0.176 + 0.058) = – 0.826 KN–m

Mba = 7 (– 2 × 0.176 – 0 + 0.058) = – 2.059 KN–m

Mbc = 5.1 + 12 (– 2 × 0.176 + 0.0986) = + 2.059 KN–m

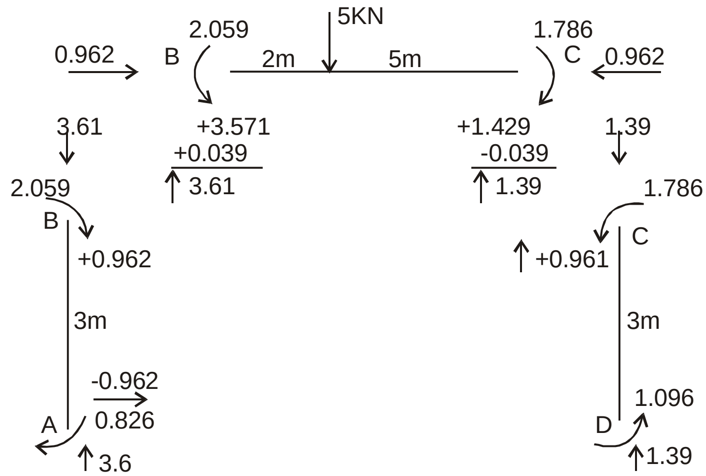
Mcb = – 2.04 + 12 (+ 2 × 0.0986 – 0.176) = – 1.786 KN–m

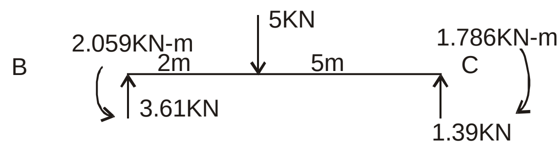
Mcd = 7 (+ 2 × 0.0986 – 0 + 0.058) = + 1.786 KN–m

Mdc = 7 (0 + 0.0986 + 0.058) = + 1.096 KN–m

Draw SFD , BMD and sketch elastic curve.

SHEAR FORCE & B.M. DIAGRAMS :– By Parts



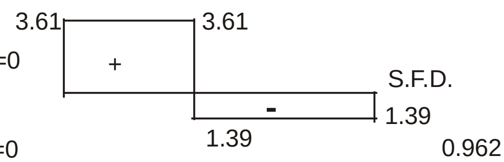


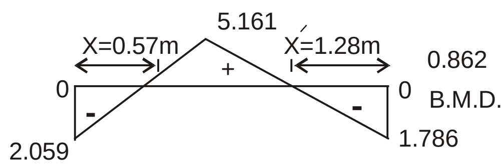
$M_x = -2.059 + 3.61X = 0$

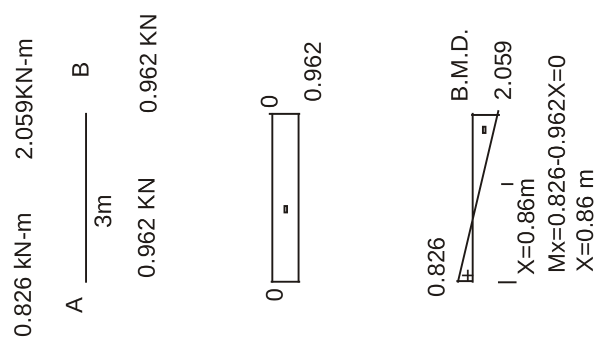
$X = 0.57m$

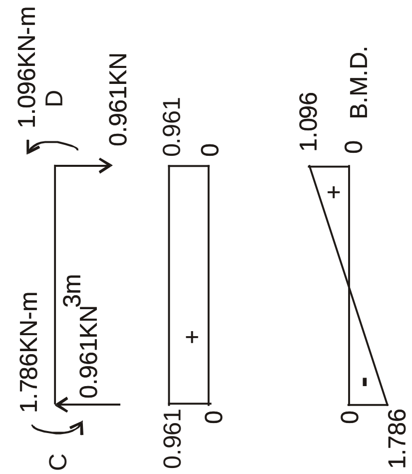
$M_x = -1.786 + 1.39X' = 0$

$X' = 1.28m$

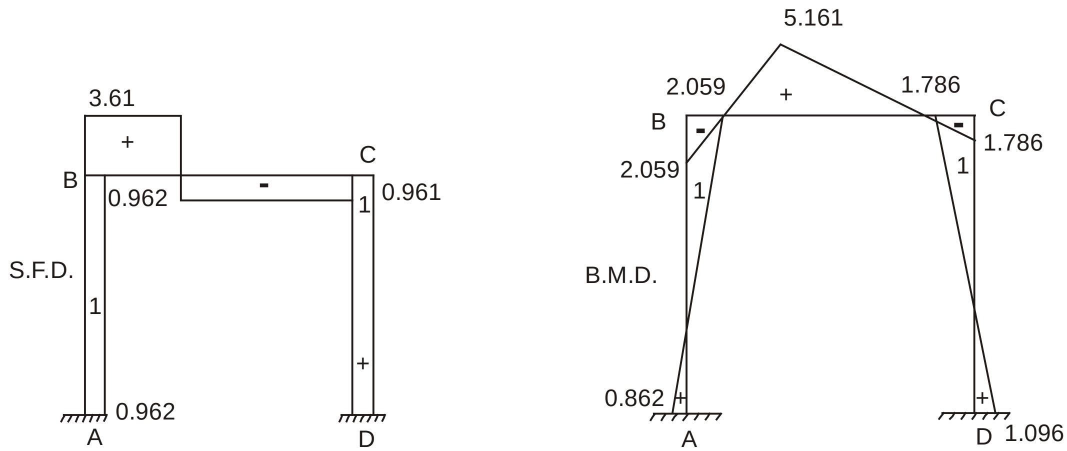






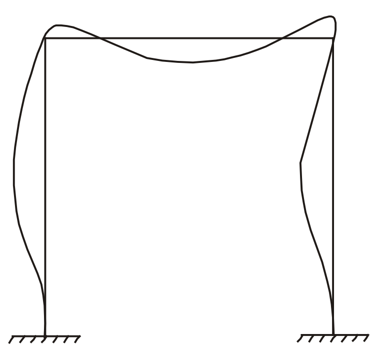


Super impositing member SFD's and BMD's.

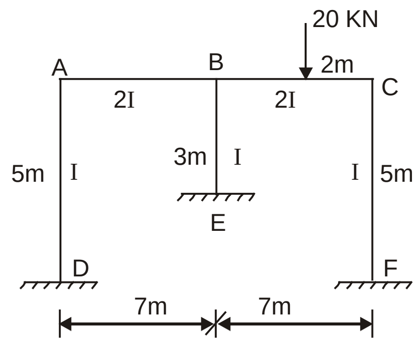


SFD **BMD**

ELASTIC CURVE:-



EXAMPLE NO. 11:– Analyze the rigid frame shown by slope–deflection method.



SOLUTION:–

Step 1: **FIXED END MOMENTS :–**

$$Mfbc = \frac{20 \times 2^2 \times 5}{7^2} = +18.16 \text{ KN–m}$$

$$Mfcb = \frac{20 \times 5^2 \times 2}{7^2} = -20.41 \text{ KN–m}$$

Mfad = Mfda = 0 |
Mfbe = Mfeb = 0 |
Mfab = Mfba = 0 | As there are no loads on these spans.
Mfcf = Mffc = 0 |

Step 2: **RELATIVE STIFFNESS:–**

Member	I	L	$\frac{I}{L}$	Krel.
AB	2	7	$\frac{2}{7} \times 105$	30
BC	2	7	$\frac{2}{7} \times 105$	30
AD	1	5	$\frac{1}{5} \times 105$	21
BE	1	3	$\frac{1}{3} \times 105$	35
CF	1	5	$\frac{1}{5} \times 105$	21

Step 3: **RELATIVE VALUES OF R :–**

Member	Δ	L	$\frac{\Delta}{L}$	Rrel.
AB	0	7	0	0
BC	0	7	0	0
AD	Δ	5	$\frac{\Delta}{5} \times 15$	3 R
BE	Δ	3	$\frac{\Delta}{3} \times 15$	5 R
CF	Δ	5	$\frac{\Delta}{5} \times 15$	3 R

Step 4: **SLOPE–DEFLECTION EQUATIONS :–**
Putting the values of fixed end moments.

Mab = 0 + 30 (– 2 θa – θb) = – 60 θa – 30 θb

Mba = 0 + 30 (– 2 θb – θa) = – 60 θb – 30 θa

Mbc = 8.16 + 30 (– 2 θb – θc) = 8.16 – 60 θb – 30 θc

Mcb = – 20.41 + 30(– 2 θc – θb) = – 20.41–60 θc–30 θb

Mad = 0 + 21 (– 2 θa – θd + 3 R) = – 42 θa + 63 R

Mda = 0 + 21 (– 2 θd – θa + 3 R) = – 21 θa + 63 R

Mbe = 0 + 35 (– 2 θb – θe + 5 R) = – 70 θb + 175 R

Meb = 0 + 35 (– 2 θe – θb + 5 R) = – 35 θb + 175 R

Mcf = 0 + 21 (– 2 θc – θf + 3 R) = – 42 θc + 63 R

Mfc = 0 + 21 (– 2 θf – θc + 3 R) = – 21 θc + 63 R

Step 5: **JOINT CONDITIONS :–**

Joint A : Mad + Mab = 0 (Continuous joint) → (1)

Joint B : Mba + Mbc + Mbe = 0 (Continuous joint) → (2)

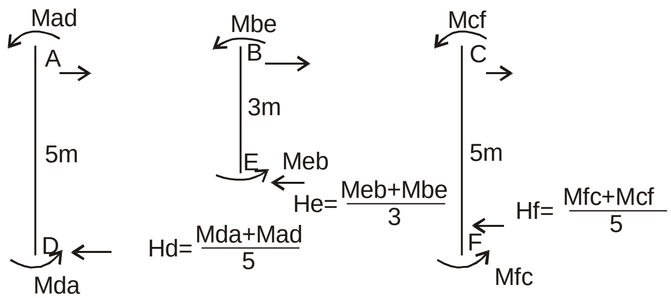
Joint C : Mcb + Mcf = 0 (Continuous joint) → (3)

Joint D : θd = 0 (Fixed end)

Joint E : θe = 0 (Fixed end)

Joint F : θf = 0 (Fixed end)

Step 6: **SHEAR CONDITIONS :–**



$\sum$ FX = 0

Hd + He + Hf = 0, Now put Hd, He and Hf in terms of end moments. We have

$$\frac{Mda + Mad}{5} + \frac{Meb + Mbe}{3} + \frac{Mfc + Mcf}{5} = 0$$

or 3 Mda + 3 Mad + 5 Meb + 5 Mbe + 3 Mfc + 3 Mcf = 0 → (4)

Step 7: **SIMULTANEOUS EQUATIONS :–**

Putting end conditions in above four equations. We have

(Mad + Mab = 0)

so $-42\,\theta a + 63\,R - 60\,\theta a - 30\,\theta b = 0$

$-102\,\theta a - 30\,\theta b + 63\,R = 0$ → (1)

Mba + Mbc + Mbe = 0

so $-60\,\theta b - 30\,\theta a + 8.16 - 60\,\theta b - 30\,\theta c - 70\,\theta b + 175\,R = 0$

$-30\,\theta a - 190\,\theta b - 30\,\theta c + 175\,R + 8.16 = 0$ → (2)

Mcb + Mcf = 0

so $-20.41 - 60\,\theta c - 30\,\theta b - 42\,\theta c + 63\,R = 0$

$-30\,\theta b - 102\,\theta c + 63\,R - 20.41 = 0$ → (3)

3Mda + 3Mad + 5Meb + 5Mbe + 3Mfc + 3Mcf = 0

so, $3(-21\,\theta a+63\,R)+3(-42\,\theta a+63\,R)+5(-35\,\theta b+175\,R - 70\,\theta b+175\,R)+3(-21\,\theta c+63\,R - 42\,\theta c+63\,R)=0$

$-63\,\theta a + 189\,R - 126\,\theta a + 189\,R - 175\,\theta b + 875\,R - 350\theta b + 875\,R - 63\,\theta c + 189\,R - 126\,\theta c + 189\,R = 0$

$-189\,\theta a - 525\,\theta b - 189\,\theta c + 2506\,R = 0$ → (4)

(not a necessary step). Writing in a matrix form to show that slope-deflection method is a stiffness method. We get a symmetric matrix about leading diagonal.

$-102\,\theta a - 30\,\theta b + 0 + 63\,R + 0 = 0$

$-30\,\theta a - 190\,\theta b - 30\,\theta c + 175\,R + 8.16 = 0$

$0 - 30\,\theta b - 102\theta c + 63\,R - 20.41 = 0$

$-189\,\theta a - 525\,\theta b - 189\theta c + 2506\,R = 0$

$-102\,\theta a - 30\theta b + 63\,R = 0 \quad \rightarrow (1)$

$-30\theta a - 190\theta b - 30\theta c + 175\,R + 8.116 = 0 \quad \rightarrow (2)$

$-30\theta b - 102\theta c + 63\,R - 20.41 = 0 \quad \rightarrow (3)$

$-189\theta a - 525\theta b - 189\theta c + 2506\,R = 0 \quad (4)$

Solve the above equations, find end moments and hence draw, S.F, B.M, elastic curse diagrams.
Solving aboving 4 equations, following values, are obtained.
$\theta a = -0.024924$, $\theta b = 0.0806095$, $\theta c = -0.225801$, $R = -0.00196765$.(use programmable calculator or Gausian elimination)
Putting these values in step 4, nodal moments may be calculated as follows:

$Mab = 0 + 30\,(-2\theta a - \theta b) = -60\theta a - 30\theta b$
$= -60(-0.024924) - 30\,(0.0806095)$
$= 0.923$ KN-m.

$Mba = -60\theta b - 30\theta a = -60(.0806095) - 30(-0.024924) = -4.089$ KN-m.

$Mbc = 8.16 - 60\,(.0806095) - 30\,(-0.225801) = 10.097$ KN-m.

$Mcb = -20.41 - 60\,(-.225801) - 30\,(0.0806095) = 0.928$ KN-m.

$Mad = -42\,(-.024924) + 63\,(-.00196765) = 0.923$ KN-m.

$Mda = -21(-.024924) + 63\,(-.00196765) = 0.3994$ KN-m.

$Mbe = -70\,(.0806095) + 175\,(-.00196765) = -5.987$ KN-m.

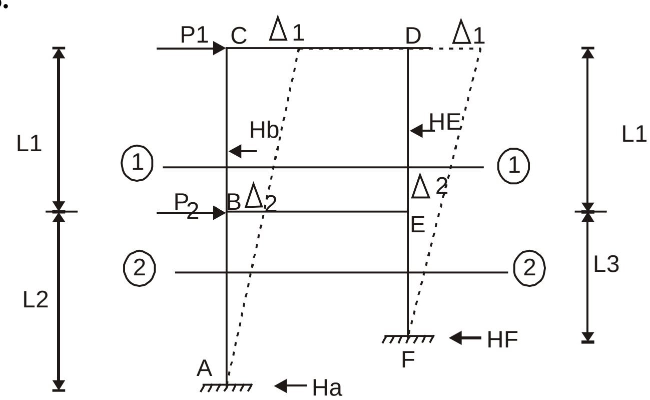
$Meb = -35(0.0806095) + 175\,(-.00196765) = 3.166$ KN-m.

$Mef = -42(0.225801) + 63\,(-.001968) = -9.60$ KN-m.

$Mfc = -21\,(-0.2258) + 63\,(-.00197) = 4.12$ KN-m.

SFD, BMD and elastic curve can be sketched now as usual.

4.9. DOUBLE STOREYED FRAMES WITH SIDE SWAY(GENERALIZED TREATMENT) FOR R VALUES.



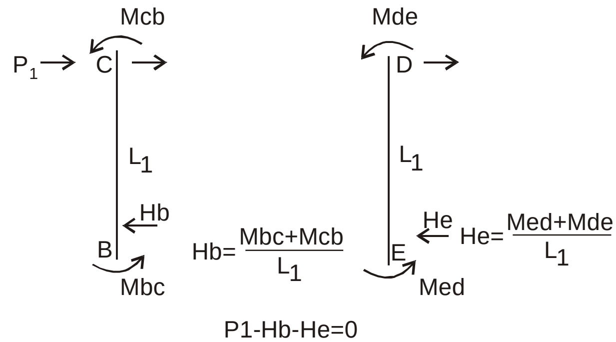
$Rbc = Red = \dfrac{\Delta_1 - \Delta_2}{L_1}$

$Rab = \dfrac{\Delta_2}{L_2} \qquad Ref = \dfrac{\Delta_2}{L_3}$

If $L_1 = L_3$

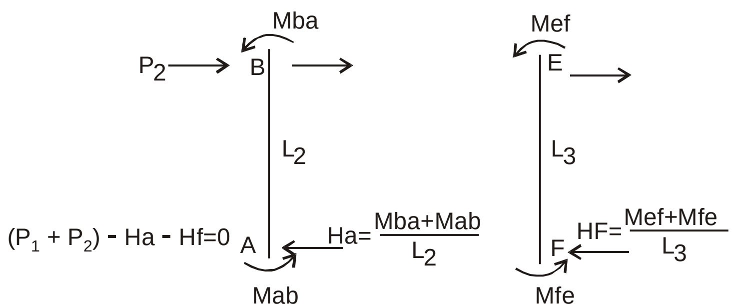
Then $Rab = \dfrac{\Delta_2}{L_2}$, $Ref = \dfrac{\Delta_2}{L_1}$

4.9.1. SHEAR CONDITIONS FOR UPPER STOREY :–



ΣFX = 0 Hb and He can be written in terms of end moments as above. Applied load upto Section-1-1.

4.9.2. SHEAR CONDITIONS FOR LOWER STOREY :–



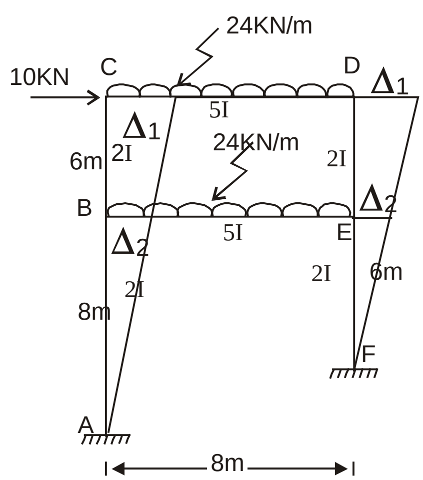
ΣFX = 0 Applied shear is to be considered upto Section 2-2. To demonstrate the application, let us solve the following question.

EXAMPLE NO. 12:- Analyze the following frame by slope – deflection method. Consider:

$I = 500 \times 10^{-6} m^4$,

$E = 200 \times 10^6 KN/m^2$

It is a double story frame carrying gravity and lateral loads.



SOLUTION :–

Step 1: **Relative Stiffness:–**

Member	I	L	$\frac{I}{L}$	K_{rel}
AB	2	8	$\frac{2}{8} \times 24$	6
BC	2	6	$\frac{2}{6} \times 24$	8
CD	5	8	$\frac{5}{8} \times 24$	15
DE	2	6	$\frac{2}{6} \times 24$	8
EF	2	6	$\frac{2}{6} \times 24$	8
BE	5	8	$\frac{5}{8} \times 24$	15

Step 2: **Relative Values of R.**

For upper story columns

$$Rbc = Rde = \frac{\Delta_1 - \Delta_2}{6} = R_1 \quad \text{(Say)}$$

$$Rab = \frac{\Delta_2}{8} \times 24 \qquad Ref = \frac{\Delta_2}{6} \times 24$$

$$Rab = 3\,R_2 \quad \text{(say)} \qquad Ref = 4\,R_2 \quad \text{(Say)}$$

Because lower story columns have different heights.

Step 3: **F.E.M :–**

F.E.M.s are induced in beams only as no loads act within column heights.

$$Mfbe = Mfcd = \frac{24 \times 8^2}{12} = +\,128 \text{ KN–m}$$

$$Mfeb = Mfdc = -\,128 \text{ KN–m}$$

Step 4: **Slope – Deflection Equations :–** Put values of FEM's and R Values for columns.

$$M_{AB} = 0 + 6\,(-2\theta a - \theta b + 3\,R_2)$$

$$M_{BA} = 0 + 6\,(-\,2\theta b - \theta ac + 3\,R_2)$$

$$M_{BC} = 0 + 8\,(-2\theta b - \theta c + R_1)$$

$M_{CB} = 0 + 8\,(-2\theta c - \theta b + R_1)$

$M_{CD} = 128 + 15\,(-2\theta c - \theta d)$

$M_{DC} = -128 + 15\,(-2\theta d - \theta c)$

$M_{DE} = 0 + 8\,(-2\theta d - \theta e + R_1)$

$M_{ED} = 0 + 8\,(-2\theta e - \theta d + R_1)$

$M_{EF} = 0 + 8\,(-2\theta e - \theta f + 4R_2)$

$M_{FE} = 0 + 8\,(-2\theta f - \theta e + 4R_2)$

$M_{BE} = 128 + 15\,(-2\theta b - \theta e)$

$M_{EB} = -128 + 15\,(-2\theta e - \theta b)$

Step 5: **Joint Conditions :–**

Joint A: $\theta a = 0$ (Fixed joint)

Joint B: $M_{BA} + M_{BC} + M_{BE} = 0$ → (1)

Joint C: $M_{CB} + M_{CD} = 0$ → (2)

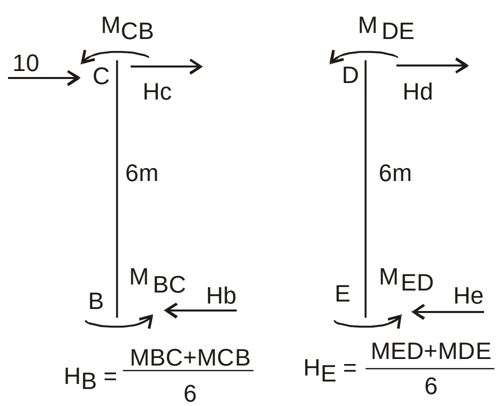
Joint D: $M_{DC} + M_{DE} = 0$ → (3)

Joint E: $M_{ED} + M_{EB} + M_{EF} = 0$ → (4)

Joint F: $\theta_f = 0$ (Fixed joint)

Step 6: **Shear Conditions :–**

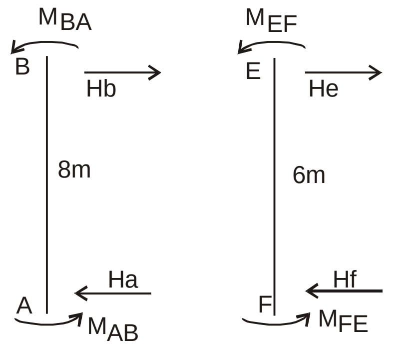
For Upper Storey :–



ΣFX = 0, 10 – Hb –He =0 putting values of Hb and He interms of end moments and simplifying, we get.

$60 - M_{BC} - M_{CB} - M_{ED} - M_{DE} = 0$ → (5)

For Lower Storey.



$H_A = \frac{M_{AB}+M_{BA}}{8}$ $H_F = \frac{M_{FE}+M_{EF}}{6}$

Σ FX = 0, 10 – Ha – Hf = 0

Putting the values of Ha and Hf in terms of end moments and simplifying, we get.

$480 - 6\,M_{AB} - 6\,M_{BA} - 8\,M_{FE} - 8\,M_{EF} = 0$ → (6)

Now we have got six equations and Six unknowns. (θb, θc, θd, θe, R_1, R_2)

Step 7: **Simultaneous Equations :–**

Putting joint conditions in slope deflection equations we have.

Mba + Mbc + Mbe = 0, $-12\theta_B+18R_2-16\theta_B-8\theta_C+8R_1+128-30\theta_B-15\theta_E = 0$

or $-58\theta_B - 8\,\theta_C - 15\theta_E + 8R_1 + 18R_2 + 128 = 0$ → (1)

Mcb + Mcd = 0

$-16\theta_C - 8\theta_B + 8R_1 + 128 - 30\theta_C - 15\theta_D = 0$

or $-8\theta_B - 46\theta_C - 15\theta_D + 8R_1 + 128 = 0$ → (2)

Mdc + Mde = 0

$-128 - 30\theta_D - 15\theta_C - 16\theta_D - 8\theta_E + 8\,R_1 = 0$

or $-15\theta_C - 46\theta_D - 8\theta_E + 8\,R_1 - 128 = 0$ → (3)

Med + Meb + Mef = 0

$-16\theta_E - 8\theta_D + 8\,R_1 - 128 - 30\theta_E - 15\theta_B - 16\theta_E + 32R_2 = 0$

or $-15\theta_B - 8\theta_D - 62\theta_E + 8\,R_1 + 32R_2 - 128 = 0$ → (4)

Putting expressions of end moments in equations 5 and 6 , we have.

$60 - (-16\theta_B - 8\theta_C + 8\,R_1 - 16\theta_C - 8\theta_B + 8R_1) - (-16\theta_E - 8\theta_D + 8R_1 - 16\theta_D - 8\theta_E + 8R_1) = 0$

or $60 + 16\theta_B + 8\theta_C - 8R_1 + 16\theta_C + 8\theta_B - 8\,R_1 + 16\theta_E + 8\theta_D - 8R_1 + 16\theta_D + 8\theta_E - 8R_1 = 0$

or $24\theta_B + 24\theta_C + 24\theta_D + 24\theta_E - 32R_1 + 60 = 0$ → (5)

$480 - 6(-6\theta_B + 18R_2 - 12\theta_B + 18R_2) - 8(-16\theta_E + 32R_2 - 8\theta_E + 32R_2) = 0$

or $480 + 108\theta_B - 216\,R_2 + 192\,\theta_E - 512\,R_2 = 0$

or $108\,\theta_B + 192\,\theta_E - 728\,R_2 + 480 = 0$ → (6)

Solving above six equations, we have.

θb=2.721 rad, θc=3.933 rad, θd= –3.225 rad, θe= –1.545 rad, R1=3.289 rad, R2=0.656 rad.

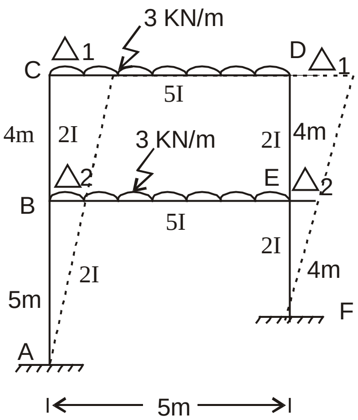
Putting these in slope deflection equations, the values of end moments are.

Mab= –4.518, Mba= –20.844, Mbc= –48.688. Mcb= –58.384, Mcd=58.384, Mdc=–90.245, Mde=90.272, Med= 76.816, Mef=45.696, Mfe=33.344 , Mbe= 69.53, Meb = –122.495 KN-m

Now SFD, BMD and elastic curve can be sketched as usual.

EXAMPLE NO. 13:– Analyze the rigid frame shown by slope–deflection method.

SOLUTION: It is a double storey frame carrying gravity loads only. Because of difference in column heights, it has become an unsymmetrical frame.



Step 1: **RELATIVE STIFFNESS.**

Member	I	L	$\frac{I}{L}$	Krel
AB	2	5	$\frac{4}{5} \times 10$	8
BC	2	4	$\frac{2}{4} \times 10$	5
CD	5	5	$\frac{5}{5} \times 10$	10
DE	2	4	$\frac{2}{4} \times 10$	5
Ef	2	4	$\frac{2}{4} \times 10$	5
BE	5	5	$\frac{5}{5} \times 10$	10

Step 2: **F.E.M :–**

F.E.Ms. are induced in beams only as they carry u.d.l. No loads act within column heights.

$$Mfbe = Mfcd = \frac{3 \times 25}{12} = +\,6.25 \text{ KN–m}$$

Mfeb = Mfdc = – 6.25 KN–m.

Step 3: **RELATIVE VALUES OF R :–**

Member	Δ	L	$\frac{\Delta}{L}$	Krel
AB	Δ_2	5	$\frac{\Delta_2}{5} \times 20$	$4 R_2$
BC	$(\Delta_1 – \Delta_2)$	4	$\frac{\Delta_1 - \Delta_2}{4}$	R_1
CD	0	5	0	0
DE	$(\Delta_1 – \Delta_2)$	4	$\frac{\Delta_1 - \Delta_2}{4}$	R_1
EF	Δ_2	4	$\frac{\Delta_2}{4} \times 20$	$5 R_2$
BE	0	5	0	0

Δ_2 terms have been arbitrarily multiplied by 20 while $\frac{\Delta_1 - \Delta_2}{4}$ has been taken equal to R_1.

Step 4: **SLOPE – DEFLECTION EQUATIONS :–**

By putting FEM's and Krel Values.

$M_{ab} = 0 + 8 (- 2 \theta a - \theta b + 4 R_2) = - 8 \theta b + 32 R_2$

$M_{ba} = 0 + 8 (- 2 \theta b - \theta a + 4 R_2) = - 16 \theta b + 32 R_2$

$M_{bc} = 0 + 5 (- 2 \theta b - \theta c + R_1) = - 10 \theta b - 5 \theta c + 5 R_1$

$M_{cb} = 0 + 5 (- 2 \theta c - \theta b + R_1) = - 10 \theta c - 5 \theta b + 5 R_1$

$M_{cd} = 6.25 + 10 (- 2 \theta c - \theta d) = 6.25 - 20 \theta c - 10 \theta d$

$M_{dc} = - 6.25 + 10 (- 2 \theta d - \theta c) = - 6.25 - 20 \theta d - 10 \theta c$

$M_{de} = 0 + 5 (- 2 \theta d - \theta e + R_1) = - 10 \theta d - 5 \theta e + 5 R_1$

$M_{ed} = 0 + 5 (- 2 \theta e - \theta d + R_1) = - 10 \theta e - 5 \theta d + 5 R_1$

$M_{ef} = 0 + 5 (- 2 \theta e - \theta f + 5 R_2) = - 10 \theta e + 25 R_2$

$M_{fe} = 0 + 5 (- 2 \theta f - \theta e + 5 R_2) = - 5 \theta e + 25 R_2$

$M_{be} = 6.25 + 10 (- 2 \theta b - \theta e) = 6.25 - 20 \theta b - 10 \theta e$

$M_{eb} = - 6.25 + 10 (- 2 \theta e - \theta b) = - 6.25 - 20 \theta e - 10 \theta b$

Step 5: **JOINT CONDITIONS :–**

Joint A : $\theta a = 0$ (Fixed joint)

Joint B : $M_{ba} + M_{bc} + M_{be} = 0$ $\rightarrow$ (1)

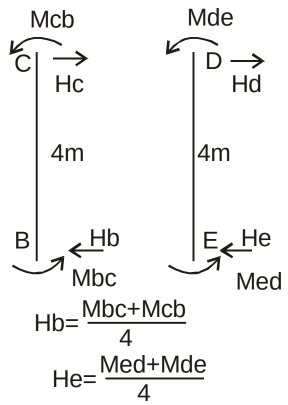
Joint C : Mcb + Mcd = 0 → (2)

Joint D : Mdc + Mde = 0 → (3)

Joint E : Med + Meb + Mef = 0 → (4)

Joint F : θf = 0 (Fixed joint)

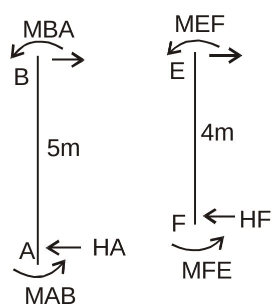
Step 6: **SHEAR CONDITIONS :–** Upper Storey



ΣFX = 0, Hb + He = 0 , Now putting their values → (5)

$$\left(\frac{Mbc + Mcb}{4}\right) + \left(\frac{Med + Mde}{4}\right) = 0 \qquad \text{Simplify}$$

Mbc + Mcb + Med + Mde = 0 → (5)



Shear Condition: Lower Storey.

ΣFX = 0, Ha + Hf = 0

$$Ha = \left(\frac{Mab + Mba}{5}\right), \; Hf = \left(\frac{Mfe+Mef}{4}\right) \qquad \text{Simplify}$$

4 (Mab + Mba) + 5 (Mfe + Mef) = 0 → (6)

Step 7: **SIMULTANEOUS EQUATIONS :–**

Putting joint and shear conditions in above six equations and simplify.

$Mba + Mbc + Mbe = 0$

or $-16\,\theta b + 32\,R_2 - 10\,\theta b - 5\,\theta c + 5\,R_1 + 6.25 - 20\,\theta b - 10\theta e = 0$

$-46\,\theta b - 5\,\theta c - 10\,\theta e + 5\,R_1 + 32\,R_2 + 6.25 = 0 \quad \rightarrow (1)$

$Mcb + Mcd = 0$

or $-10\,\theta c - 5\,\theta b + 5\,R_1 + 6.25 - 20\,\theta c - 10\,\theta d = 0$

$-5\,\theta b - 30\,\theta c - 10\,\theta d + 5\,R_1 + 6.25 = 0 \quad \rightarrow (2)$

$Mdc + Mde = 0$

or $-6.25 - 5\,\theta d - 10\,\theta c - 10\,\theta d - 5\,\theta e + 5\,R_1 = 0$

$-10\,\theta c - 30\,\theta d - 5\,\theta e + 5\,R_1 - 6.25 = 0 \quad \rightarrow (3)$

$Med + Mcb + Mef = 0$

or $-10\,\theta e - 5\,\theta d + 5\,R_1 - 6.25 - 20\,\theta e - 10\,\theta b - 10\,\theta e + 25\,R_2 = 0$

$-10\,\theta b - 5\,\theta d - 40\,\theta e + 5\,R_1 + 25\,R_2 - 6.25 = 0 \quad \rightarrow (4)$

$Mbc + Mcb + Med + Mde = 0$

or $-10\,\theta b - 5\,\theta c + 5\,R_1 - 10\,\theta c - 5\,\theta b + 5\,R_1 - 10\,\theta d$

$-5\,\theta e + 5\,R_1 - 10\,\theta e - 5\,\theta d + 5\,R_1 = 0$

$-15\,\theta b - 15\,\theta c - 15\,\theta d - 15\,\theta e + 20\,R_1 = 0 \quad \rightarrow (5)$

$4(Mab + Mba) + 5\,(Mfe + Mef) = 0$

or $4\,(-8\,\theta b + 32\,R_2 - 16\,\theta b + 32\,R_2) + 5(-10\,\theta e + 25\,R_2 - 5\,\theta e + 25\,R_2) = 0$

$-96\,\theta b - 75\,\theta e + 506\,R_2 = 0 \quad \rightarrow (6)$

Solving above six equations (by programmable calculator) we have.

$\theta b=0.141$, $\theta c=0.275$, $\theta d= -0.276$, $\theta e= -0.156$, R1=0.01224, R2=0.003613.
By Putting these in slope deflection equations, the values of end moments are.

Mab = –1.012, Mba = –2.14, Mbc = –2.846, Mcb = –3.5162, Mcd = 3.51, Mdc = –3.48, Mde = 3.52, Med = 2.8788, Mef = 1.65, Mfe = 0.87, Mbe = 4.99, Meb = –4.54

Now SFD, BMD and elastic curve can be sketched as usual.