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Essays on automorphic forms

Stereographic projection

Bill Casselman
University of British Columbia
cass@math.ubc.ca

It is impossible to render paths on a sphere onto a flat surface in such a way that all distances remain the same. In drawing a map of a sphere, therefore, some compromises must be made. Most maps adopt one of two possible strategies—that areas are preserved or that angles are preserved. **Stereographic projection** is one way of making maps, and it preserves angles. It has been used since ancient times for this purpose, and its basic geometrical properties were known even then.

The main results of this chapter are that:

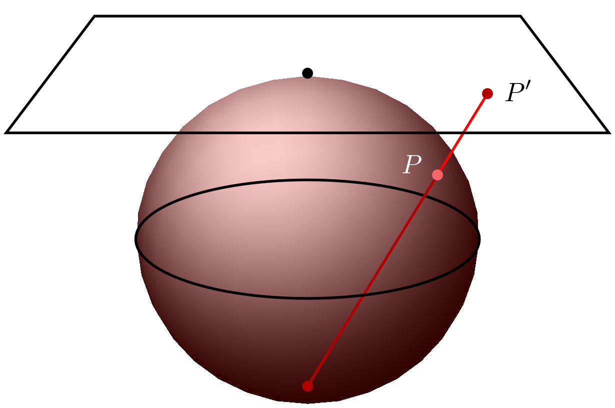
- *stereographic projection takes circles to circles;*
- *it is conformal;*
- *it endows the sphere* $\mathbb{S}^2$ *with a complex structure.*

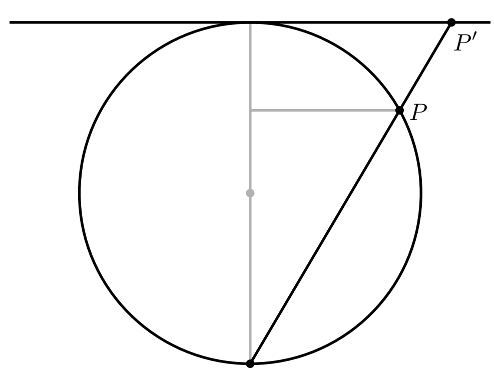
Proofs will be in the style of classical geometry.

Contents

1. The map

Consider the unit sphere $x^2 + y^2 + z^2 = 1$ in three dimensions, capped by the tangent plane $z = 1$ through the north pole. We want to define a projection from the sphere onto this plane. If P is a point on the sphere, let P' be the intersection of the ray from the south pole $\Pi = (0, 0, -1)$ to P with the plane.





This definition fails if P is Π itself. Therefore stereographic projection maps all points on the sphere except Π to a point on the polar plane, and its inverse wraps the plane around the complement of Π.

Explicitly, if $P = (x, y, z)$ with $z \neq -1$ then the parametrized line through Π and P is $P + t(\Pi - P) = (tx, ty, 1 - t - tz)$. This intersects $z = 1$ when $t = 2/(1 + z)$, which makes

$$P' = (X, Y, 0), \quad X = \frac{2x}{1+z}, \quad Y = \frac{2y}{1+z} \; .$$

The inverse takes (X, Y) to the point (x, y, z) on the unit sphere lying on the line through $(X, Y, 0)$ and Π. Thus we must solve

$$\begin{aligned} x &= sX \\ y &= sY \\ z &= 2s - 1 \end{aligned}$$

subject to

$$x^2 + y^2 + z^2 = s^2(X^2 + Y^2 + 4) - 4s + 1 = 1$$

or

$$s^2(X^2 + Y^2 + 4) - 4s = 0 \, .$$

One solution is $s = 0$, giving Π, and the other

$$s = \frac{4}{X^2 + Y^2 + 4} \, .$$

Stereographic projection is distinguished by these two related properties:

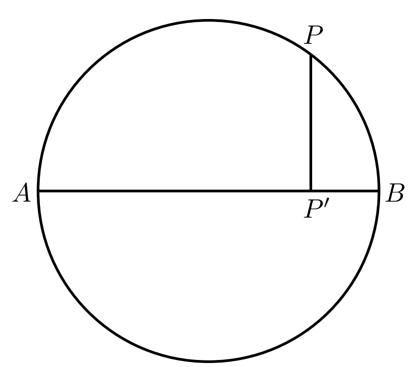
- Circles on the sphere correspond to circles on the plane, except that circles through Π correspond to lines;
- stereographic projection preserves the angle between paths (it is conformal).

The first of these was known to the Greeks of the Hellenistic age, and can be found early in Apollonius' treatise on conic sections. It was crucial in the design of astrolabes. The second seems to have been first discovered by the English mathematician Thomas Harriot, cartographer and navigator for Walter Raleigh, but his proof remained unpublished until long after his death.

I'll prove these two properties in the next few sections. First I recall some simple geometry of circles, and then prove an elementary property of cones.

2. Circles

The first result to be brought up is a charcaterization of circles that was used frequently by the ancient Greeks. It is a kind of substitute for coordinate geometry. Suppose given two points A, B. For each point P in the plane let P' be the perpendicular projection of P onto the line through A and B.



Proposition. *The circle whose diameter is AB consists of all points P such that*

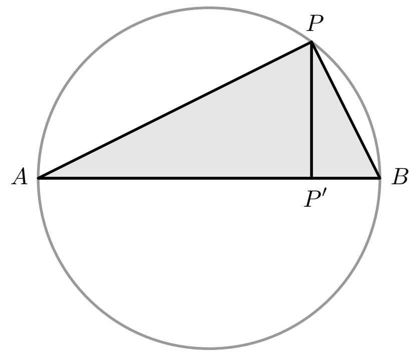
$$PP'^2 = AP' \cdot P'B\,.$$

If r is the radius of the circle, and $P = (x, y)$ then $AP' = r + x$ and $P'B = r - x$, so this is equivalent to the equation $y^2 = r^2 - x^2$.

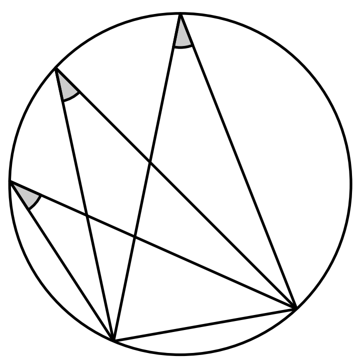
The equation above can be rewritten as

$$\frac{PP'}{AP'} = \frac{P'B}{PP'}\ .$$

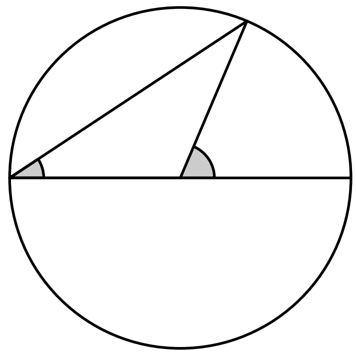
In this form, it expresses the similarity of the two triangles in the figure below.



Another result we'll need later on is that all the angles in the following picture are the same:



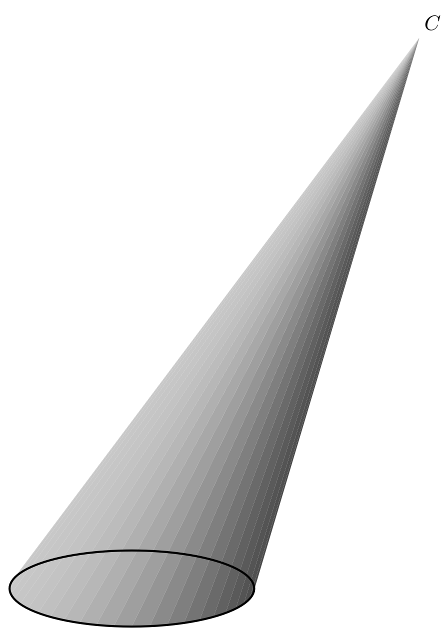
This follows in turn from the fact that the central angle in this figure is twice the exterior one.



I leave these as exercises.

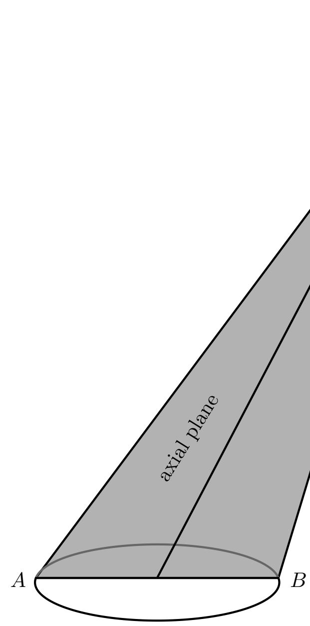
3. Circular sections of a cone

Place a circle in a plane, and choose a point C not on that plane. Construct the cone with C as vertex intersecting the plane in the given circle.

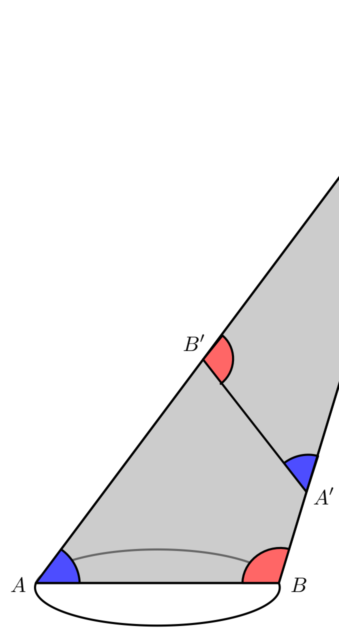


Any section of the cone parallel to the given plane is also a circle. If the cone is a **right** cone—with its axis perpendicular to the plane—it is easy to see that only horizontal slices intersect it in circles. But if it is **oblique**, then there exist other circular sections of the cone as well. In fact there exists a single parallel family of them, a family conjugate to the first. I'll follow Apollonius' treatise on conic sections (Book I, Proposition 5) in explaining how to see this.

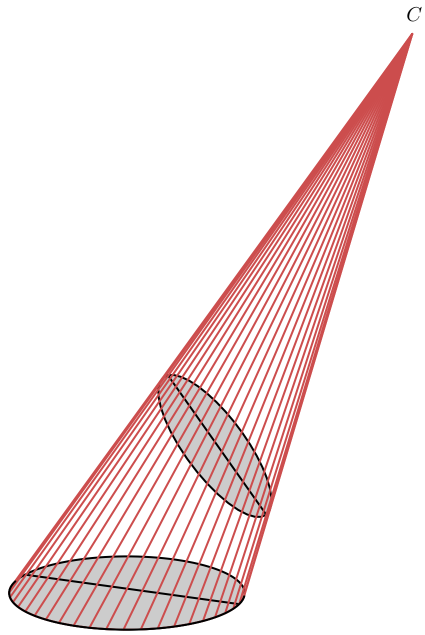
To construct one of these circles, erect first the plane perpendicular to the original one, containing the axis of the cone. This plane will contain the vertex, and will also contain a diameter AB of the original circle. I call this the **axial plane**.



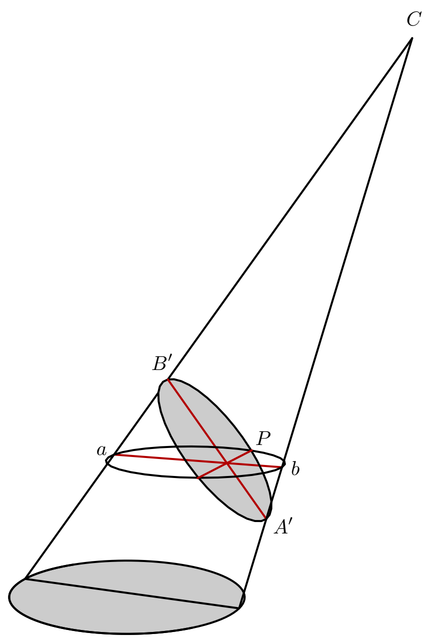
Construct inside the triangle ABC a **contrary** triangle $A'B'C$ similar to ABC, but with opposite orientation.



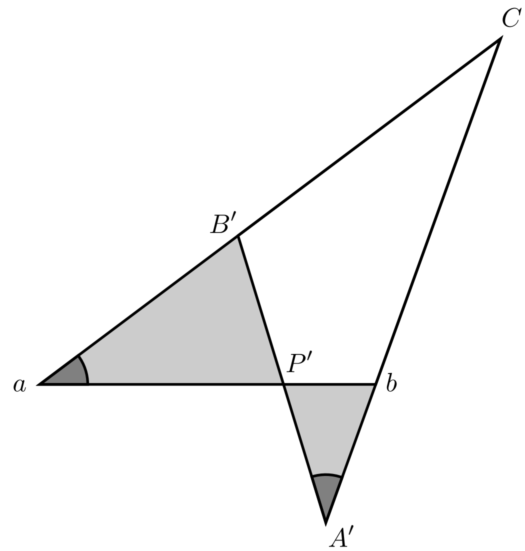
Erect a plane perpendicular to the axial plane running through $A'B'$.



We want to show that *the section of the cone by this plane is a circle.*



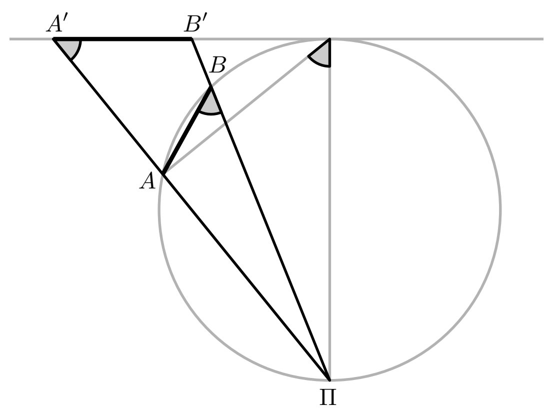
If P is any point of this intersection and P' is the foot of the perpendicular from P to $A'B'$, we must show that $PP'^2 = A'P' \cdot P'B'$. Pass a plane parallel to the original one through the line PP', and let a and b be the points on this plane corresponding to A and B. Since this section by this new plane is a circle, we know that $PP'^2 = aP' \cdot P'b$. But the following figure explains why $aP' \cdot P'b = A'P' \cdot P'B'$.



This result is one of the first results in Apollonius' book on conics, and presumably one of the earliest non-trivial results known to the Greeks about conic sections.

4. Stereographic projection maps circles to circles

The segment AB represents a circular section of the sphere, and $A'B'$ represents its image under stereographic projection. The following figure explains why the section $A'B'$ is a contrary section of the cone generated by P and the circle AB, hence a circle.

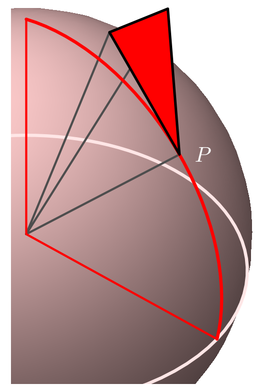


5. Conformality

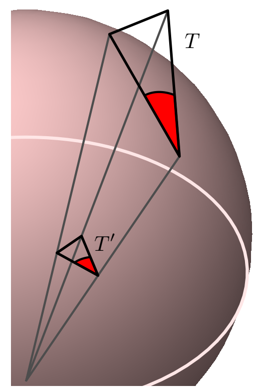
In a moment, I'll sketch a direct proof that stereographic projection is conformal, but I should point out that it follows from the previous result. That result implies that the Jacobi derivative of stereographic projection takes circles into circles. Conformality means that the action of a map on tangent spaces is a similarity transformation, one that transforms the sum of squares into some positive multiple of itself. But it is a simple exercise in linear algebra that a linear transformation that takes spheres into spheres (or, in two dimensions, circles into circles) is a similarity transformation.

Now I'll follow the original proof of Thomas Harriot. I include here just the figures necessary to trace it. We choose a point P on the sphere, and we want to show that stereographic projection preserves angles at P. It suffices to show this for meridian angles.

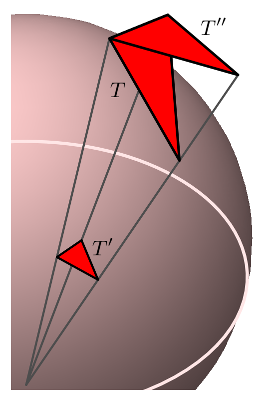
Construct a meridian triangle T at P, tangent to the unit sphere, with one right angle.



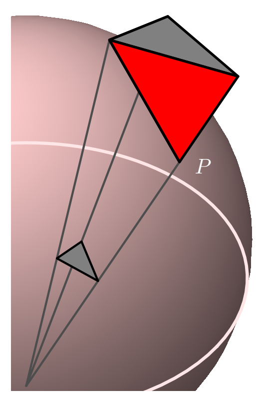
Project it onto the equatorial plane to get T', which is also right-angled. We want to show that T and T' are similar.



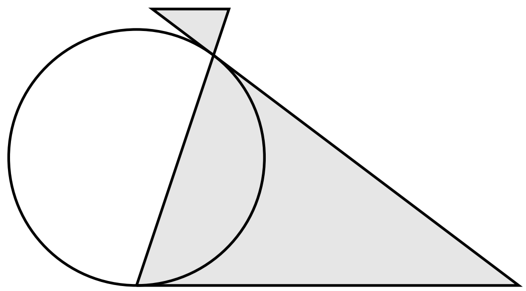
Construct a triangle T'' parallel to T', attached to T along the edge opposite P.



It is congruent to the original one, because the triangle illustrated is isosceles.



And finally a two-dimensional diagram explaining why that last triangle is indeed isosceles:



6. A complex structure on the sphere

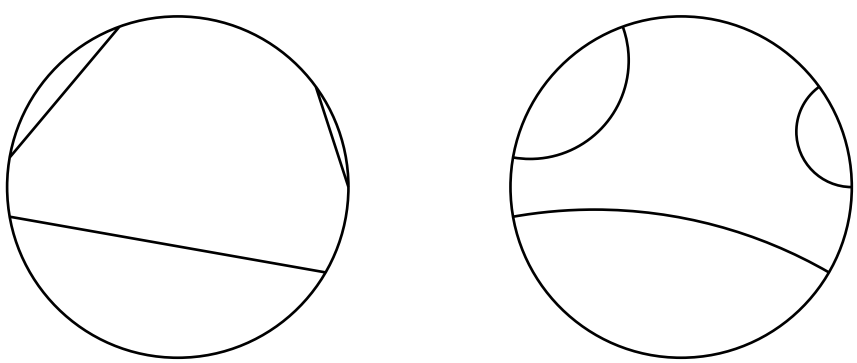
The choice of the 'south pole' $(0, 0, -1)$ as pole for projection was arbitary. If one chooses the opposite pole $(0, 0, 1)$ on obtains a second projection from the sphere onto the plane. The composition of the two takes

$$(x, y) \longmapsto (x/r^2, y/r^2)$$

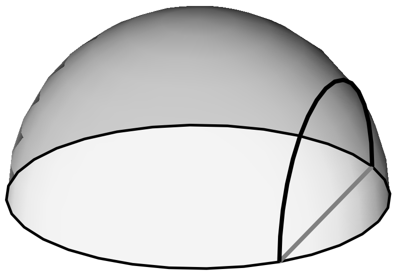
where $r^2 = x^2 + y^2$. If $z = x + iy$ this is the map taking z to $1/\overline{z}$. Conjugation is to be expected here, because the two different stereographic projectiosn differ in orientation. If we combine one of them, say that with pole $(0, 0, 1)$, with the map $(x, y) \mapsto (x, -y)$ we obtain as composite the complex map $z \mapsto 1/z$, and we have thus provided the sphere with a complex structure—each point is given locally an isomorphism with an open set in $\mathbb{C}$, and the maps on overlaps are also analytic. In this way, $\mathbb{S}^2$ becomes the **Riemann sphere**.

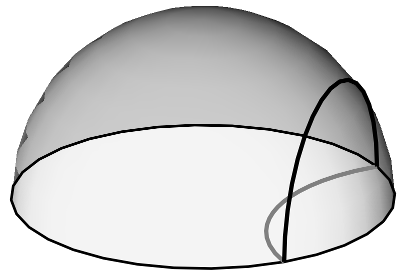
7. Klein and Poincaré

There are two well know models of non-Euclidean geometry. In both, the non-Euclidean plane is the interior D of the unit disk. In the **Klein model** the geodesics are line segmenst traversing D, while in the **Poincaré model** the geodesics are the arcs of circles in D the meet the circumference at right angles.



The transformation from the Klein model to that of Poincaré takes place in a couple of steps. The first is to project up onto the unit hemisphere, and the second is to project back to the plane by stereographic projection.





8. References

Apollonius of Perga, **Treatise on conic sections**, translated by T. L. Heath, Cambridge University Press, 1896.

D. Hilbert and S, Cohn-Vossen, **Mathematics and the imagination**, Chelsea, 1952. A very different treatment of stereographic projection can be found in §36.

J. Lohne, 'Thomas Harriot als Mathematiker', *Centaurus* **11** (1965), 19–45. Contains a facsimile of a hitherto unpublished sketch by Harriot, which in effect proves that stereographic projection is conformal.