CHAPTER NINE

9. THE THREE MOMENT EQUATION

Most of the time we are concerned with the classical analysis of statically determinate structures. In this chapter we shall consider the analysis of statically indeterminate (externally) beams due to applied loads and due to settlement of supports. It must be remembered that supports for beams may be walls or columns. As we know that for the analysis of statically indeterminate systems, compatibility of deformations is also essential requirements in addition to considerations of equilibrium and statics. By compatibility it is understood that deformations produced by applied loads should be equal to those produced by redundants. It has been already mentioned that reactions occur at supports in various directions if

(i) There is some action (applied load) in that direction.

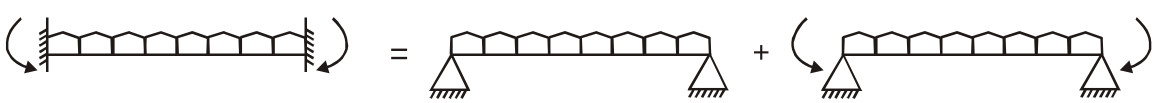
(ii) There is restraint offered by support in that directions

Action and reactions are equal in magnitude but opposite in direction. In the structural analysis it is sometimes customery to think that rotations are generally associated with moments and deflections or translations are associated with loads. It must also be kept in mind that we never analyze actual structural systems or sub-systems, it is only the idealized ones which are analyzed. Representing beams and columns by just a straight line located on their centroidal axis is also a sort of idealization on the structural geometry. Reactions and loads are, therefore, also idealized and are shown by a sort of line loads acting on a point.

The three-moment equation is a good classical analysis tool in which support moments produced by the loads as well as by the differential settlements can be easily calculated by using second-moment area theorem which states that

" The deviation of a point A on the elastic curve w.r.t any other point B on the elastic curve is equal to $\frac{1}{EI}$ multiplied by the moment of area of B.M.D's between those two points." The moments of B.M.D's are taken about a line passing through the point of loaded beam where deviation is being measured.

The method is essentially based on continuity (equality) of slopes on the either side of a support by reducing an indeterminate system to its determinate equivalents as follows by using supperposition.



An indeterminate beam under applied loads and redundant moments is equated to corresponding detemrinate system carrying these two effects separately. Let-us derive the three-moment equation.

Consider a generalized two-span beam element under the action of applied loads and redundant support moments acting on BDS.

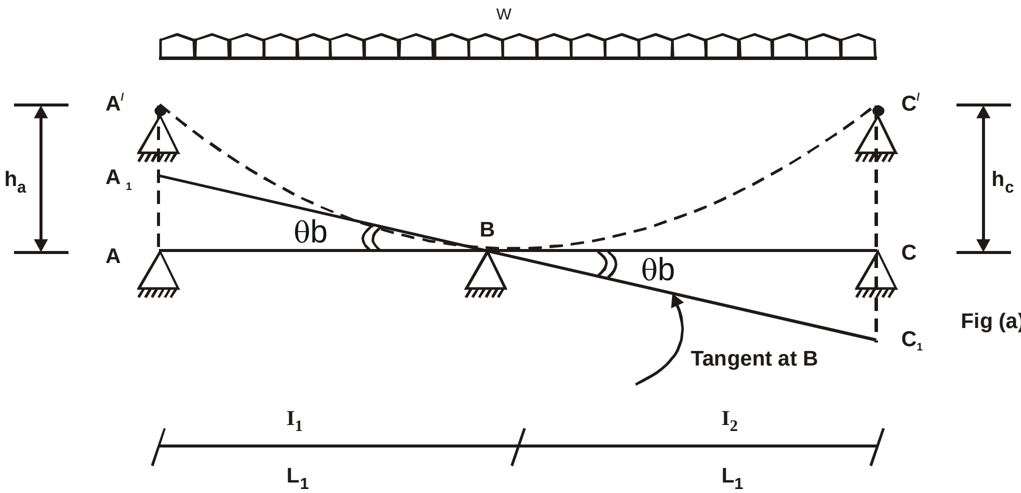


Fig (a)

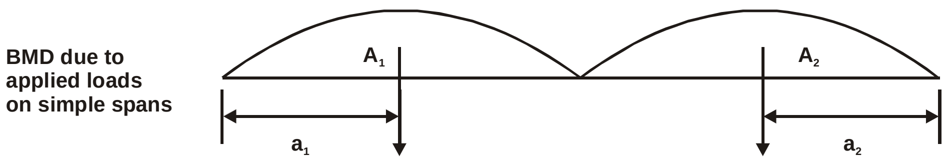


Fig (b)

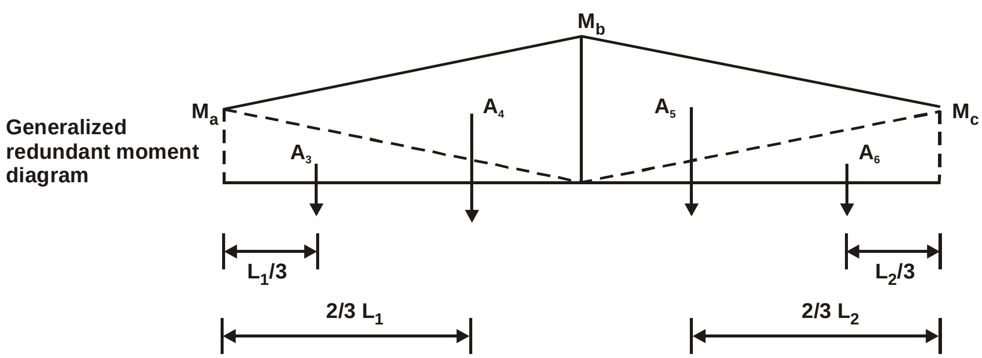


Fig (c)

Fig(a) is an indeterminate beam subjected to applied load (udl in this case) which has shown settlement such that support B is at a lower elevation than support at A and C and difference of elevation w.r.t intermediate support B is h_a and h_c. The angle θ_B on either side of support B must be equal. Fig(b) is B.M.D. due to applied load on simple spans where A_1 is Area of B.M.D. on span L_1 and A_2 is area of B.M.D. on span L_2. a_1 and a_2 are the locations of centroids of B.M.D' s on L_1 and L_2 from left and right supports respectively. So invoking continuity of slopes and knowing that for small angels $\theta = \tan\theta$.

$$\frac{AA_1}{L_1} = \frac{CC_1}{L_2}$$

Evaluate AA_1 by second Moment Area Method. We know that

$$AA_1 = AA' - A_1A'$$

$= h_a$ – deviation of point A' on the elastic curve from the tangent drawn at point B on the elastic curve.

$$= h_a - \frac{1}{EI_1}\left[A_1a_1 + A_3 \times \frac{L_1}{3} + A_4 \times \frac{2}{3}L_1\right]$$

expressing A_3 and A_4 in terms of moments

$$AA_1 = h_a - \frac{1}{EI_1}\left[A_1a_1 + \frac{L_1}{3} \times \frac{1}{2}MaL_1 + \frac{2}{3}L_1 \times \frac{1}{2}MbL_1\right]$$

$$= h_a - \frac{1}{EI_1}\left[A_1a_1 + \frac{MaL_1^2}{6} + \frac{MbL_1^2}{3}\right] \quad \text{divide by } L_1$$

$$\frac{AA_1}{L_1} = \frac{h_a}{L_1} - \frac{1}{EI_1}\left[\frac{A_1a_1}{L_1} + \frac{MaL_1}{6} + \frac{MbL_1}{3}\right] \qquad (1)$$

Now evaluate $\frac{CC_1}{L_2}$ on similar lines. We have from geometry

$$CC_1 = C_1C' - CC'$$

$= $ (deviation of point C' from tangent at B) $- h_c$

$$= \frac{1}{EI_2}\left[A_2a_2 + A_5 \times \frac{2}{3}L_2 + A_6 \times \frac{L_2}{3}\right] - h_c$$

expressing A_5 and A_6 in terms of Moments

$$CC_1 = \frac{1}{EI_2}\left[A_2a_2 + \frac{2}{3}L_2 \times \frac{1}{2}MbL_2 + \frac{L_2}{3} \times \frac{1}{2}MCL_2\right] - h_c$$

$$= \frac{1}{EI_2}\left[A_2a_2 + Mb\frac{L_2^2}{3} + MC\frac{L_2^2}{6}\right] - h_c \quad \text{divide by } L_2$$

$$\frac{CC_1}{L_2} = \frac{1}{EI_2}\left[\frac{A_2a_2}{L_2} + \frac{Mb\,L_2}{3} + \frac{MC\,L_2}{6}\right] - \frac{h_c}{L_2} \qquad (2)$$

Equating (1) and (2), we have

$$\frac{h_a}{L_1} - \frac{1}{EI_1}\left[\frac{A_1a_1}{L_1} + \frac{Ma\,L_1}{6} + \frac{Mb\,L_1}{3}\right] = \frac{1}{EI_2}\left[\frac{A_2a_2}{L_2} + \frac{Mb\,L_2}{3} + \frac{Mc\,L_2}{6}\right] - \frac{h_c}{L_2}$$

Multiply by 6E and simplify, we have after re-arrangement

$$M_a\left(\frac{L_1}{I_1}\right) + 2M_b\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + M_c\left(\frac{L_2}{I_2}\right) = -\frac{6A_1\bar{a}_1}{I_1L_1} - \frac{6A_2\bar{a}_2}{I_2L_2} + \frac{6Eh_a}{L_1} + \frac{6Eh_c}{L_2}$$

The above equation is called three-moment equation.

9.1. Analysis of Continuous Beams by three-Moment Equation.

We apply three moment equation to two spans at a time which gives us one equation. With the successive applications, the required member of equations are obtained and are solved simultaneously.

EXAMPLE: Analyze the continuous beam shown below by three-Moment equation.
Take $E = 20 \times 10^6$ KN/m^2 and $I_c = 40 \times 10^{-6}$ m^4.

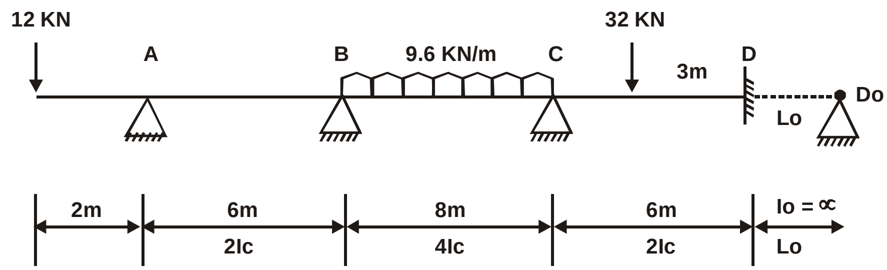


Fig (a)

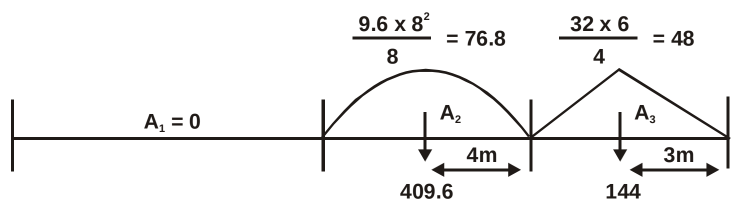


Fig (b)

SOLUTION:

When a fixed support at either end is encountered, an imaginary hinged span of length L_o and Interia $I_o = \infty$ is added to conform to acted support conditons and to make the method applicable in similar situations.

The same has already been done in Fig(a). Fig (b) is the BMD' s on simple spans, their Areas and its locations.

Apply three-moment equation to spans AB and BC at a time. We have

$$M_a\left(\frac{6}{2I_c}\right) + 2M_b\left(\frac{6}{2I_c} + \frac{8}{4I_c}\right) + M_c\left(\frac{8}{4I_c}\right) = -6 \times 0 - \frac{6 \times 409.6 \times 4}{4I_c \times 8}$$

Simplify and multiplying by Ic both sides of equation, we get.

$3M_a + 10M_b + 2M_c = -307.2$ put $M_a = -24$ KN-m

$10M_b + 2M_c = -235.2$ divide by 10

$M_b + 0.2M_c = -23.52$ (1)

Now apply three-moment equation to spans BC and CD

$$Mb\left(\frac{8}{4Ic}\right) + 2\,Mc\left(\frac{8}{4Ic} + \frac{6}{3Ic}\right) + MD\left(\frac{6}{3Ic}\right) = -\frac{6 \times 409.6 \times 4}{4Ic \times 8} - \frac{6 \times 144 \times 3}{3Ic \times 6}$$

Simplify and multiply by Ic, we have,

$2\,Mb + 8\,Mc + 2\,MD = -307.2 - 144 = -451.3$ divide by 2

$Mb + 4\,Mc + MD = -225.625$ (2)

Now apply three-moment equation to spans CD and DDo

$$Mc\left(\frac{6}{3Ic}\right) + 2\,MD\left(\frac{6}{3Ic} + \frac{Lo}{\infty}\right) + Md_o\left(\frac{Lo}{\infty}\right) = -\frac{6 \times 144 \times 3}{3Ic \times 6}$$

Simplify and multiply by Ic both sides of equation.

$2\,Mc + 4\,MD = -144$ divide by 2

$Mc + 2\,MD = -72$ (3)

We have obtained three equations from which three-Unknowns Mb, Mc and MD can be calculated.

Subtract equation (2) from (1)

$$\begin{array}{llll} & Mb + 0.2\,Mc & & = -23.52 \\ & Mb + 4\,Mc + MD & & = -225.625 \\ \hline -3.8\,Mc & & -MD & = 202.105 \end{array} \qquad (4)$$

Multiply equation (4) by (2) and add in equation (3)

$$\begin{array}{ll} -7.6\,Mc - 2MD & = 404.21 \\ Mc + 2\,MD & = -72 \\ \hline -6.6\,MC & = 332.21 \end{array}$$

So $Mc = -50.3$ KN-m

put Mc in equation (1), we get $Mb = -13.46$ KN-m

put Mc in (3), we get $MD = -10.85$ KN-m.

Finally

$Mb = -13.46$ KN-m

$Mc = -50.3$ KN-m

$MD = -10.85$ KN-m

Checks:

The above calculated values of moments are correct if they satisfy the continuity of slope requirements. Slopes at any intermediate support point can be calculated from the two adjacent spans by using conjugate beam method. While applying checks, it is assumed that reader is well conversant with the conjugate beam method. Before we could apply checks, it is necessary to plot reactant moment diagram (support-moments) to get their contribution in slope calculation. Here is the statement of conjugate beam theorem number one again.

" The shear force at any point on the conjugate beam loaded with $\frac{M}{EI}$ diagram is the slope at the corresponding point in the actual beam carrying applied loads." In applying the conjugate -beam method, we must use the original sign convention for shear force as applied in strength of Materials subject. (i.e., " left up, right-down, positive)

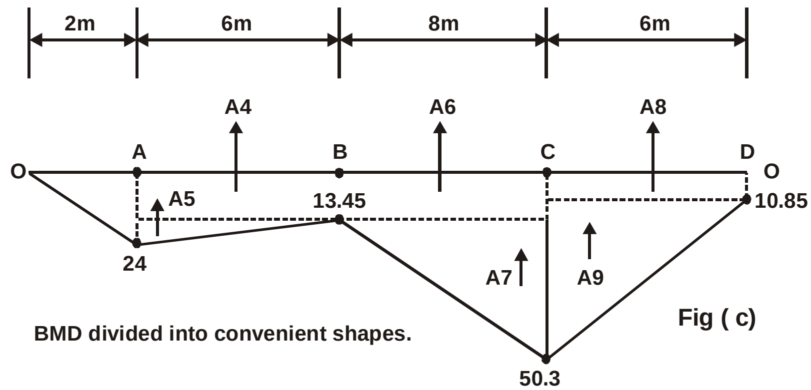


Fig (c)

Fig(c) is the reactant moment diagram

The areas of positive BMD' s act as loads in downward direction to which reactions are upwards. The areas of negative BMD' s act as loads in upward direction to which support reactions are downwards. The direction of reaction is accounted for in the signs appropriately.

$A4 = 13.45 \times 6 = 80.7$ $A7 = \dfrac{8(50.3 - 13.45)}{2} = 146.2$

$A5 = \dfrac{6(24 - 13.45)}{2} = 31.65$ $A8 = 10.85 \times 6 = 65.1$

$A6 = 13.45 \times 8 = 107.6$ $A9 = \dfrac{6(50.3 - 10.85)}{2} = 118.35$

Checks. SPAN AB

S.F at A = $\theta a = \dfrac{1}{EI}\left[-\dfrac{A4}{2} - \dfrac{2}{3}A5\right] = \dfrac{1}{2EIc}\left[\dfrac{-80.7}{2} - \dfrac{2}{3} \times 31.65\right]$

$\theta a = -\dfrac{30.725}{EIc}$ (There is no check on this value as, it is not a continuous support)

$$\theta b = \frac{1}{2EIc}\left[\frac{A4}{2} + \frac{1}{3}A5\right] = \frac{1}{2EIc}\left[\frac{80.7}{4} + \frac{31.65}{3}\right]$$

$= \dfrac{25.45}{EIc}$ Clockwise.

SPAN BC

$$\theta b = \frac{1}{4EIc}\left[\frac{A2}{2} - \frac{A6}{2} - \frac{1}{3}A7\right] = \frac{1}{4EIc}\left[\frac{409.6}{2} - \frac{107.3}{2} - \frac{1}{3} \times 147.5\right]$$

$\theta b = \dfrac{25.46}{EIc}$ Clockwise

$$\theta c = \frac{1}{4EIc}\left[\frac{-A2}{2}+\frac{A6}{2}+\frac{2}{3}A7\right] = \frac{1}{4EIc}\left[\frac{-409.6}{2}+\frac{107.3}{2}+\frac{2}{3}\times 147.5\right]$$

$$\theta c = \frac{-13.18}{EIc}$$

SPAN CD

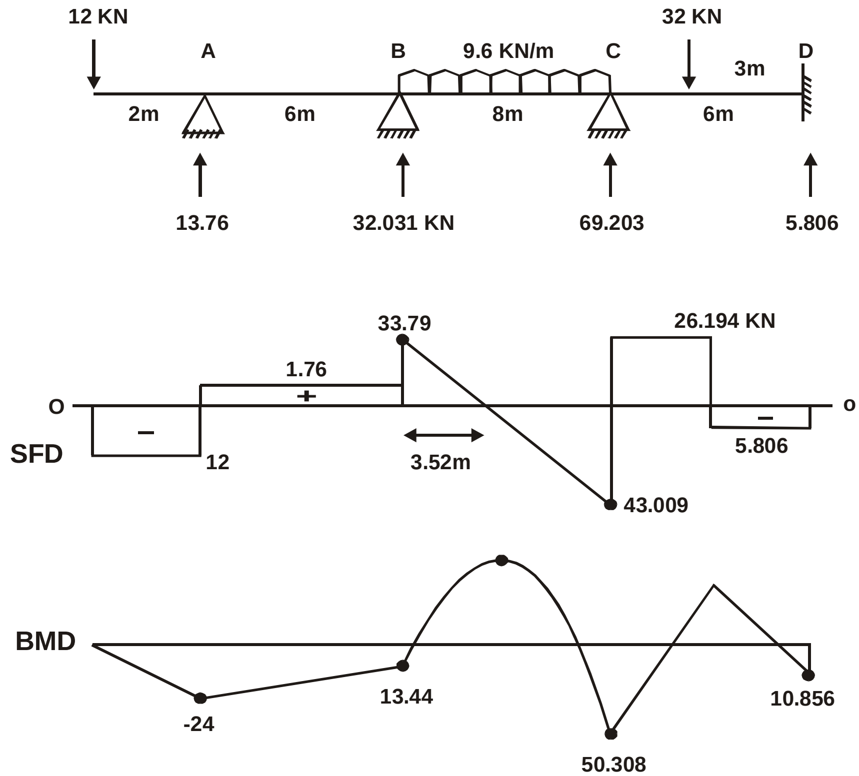
$$\theta c = \frac{1}{3EIc}\left[\frac{A3}{2}-\frac{A8}{2}-\frac{2}{3}A9\right] = \frac{1}{3EIc}\left[\frac{144}{2}-\frac{65.1}{2}-\frac{2}{3}\times 118.33\right]$$

$$\theta c = -\frac{13.16}{EIc}$$

$$\theta D = \frac{1}{3EIc}\left[-\frac{A3}{2}+\frac{A8}{2}+\frac{1}{3}A9\right] = \frac{1}{3EIc}\left[-\frac{144}{2}+\frac{65.1}{2}+\frac{118.33}{3}\right]$$

$$\theta D = 0 \quad \text{(Fixed end)}$$

All slope values have been satisfied. This means calculated support moment values are correct. Now beam is statically determinate we can construct SFD and BMD very easily. We have seen that numerical values of E and I are required in this case only if one is interested in absolute values of θ. However, these values are required while attempting a support settlement case. Determine reactions and plot SFD and BMD.



EXAMPLE-2: Analyze the continuous beam shown below by three moment equation if support at B sinks by 12 mm. Take $E = 20 \times 10^6$ KN/m^2; $Ic = 40 \times 10^{-6}$ m^4.

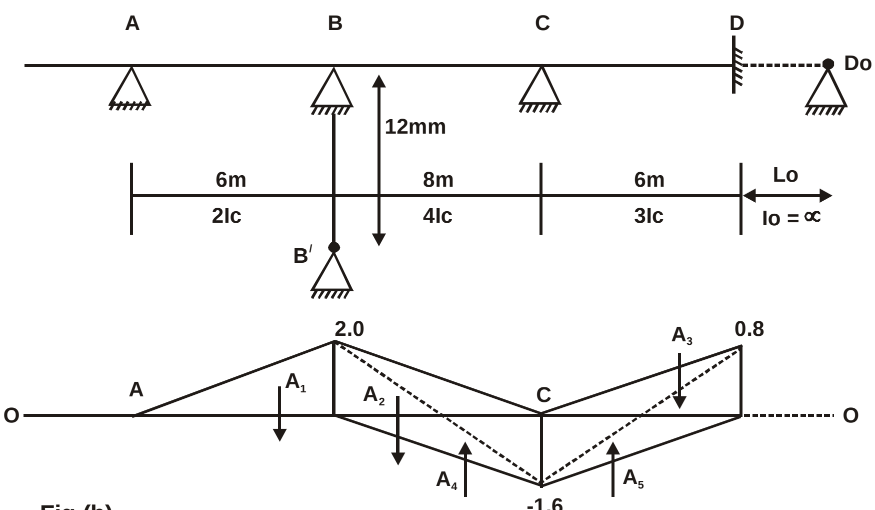


Fig (b)
Reactant moment diagram
A_1 to A_5 are areas of adjusted BMD.

SOLUTION:

As the extreme right support is fixed, an imaginary Hinged span of length Lo and Ic = ∞ has already been added to make the method applicable and to conform to the support characteristic at D. Now it is a sort of continuous support. Only analysis due to differential settlement at B is required. Had there been some applied loads also, those could have been considered at the same time also.

Now EI = $20 \times 10^6 \times 40 \times 10^{-6} = 800$ KN-m^2.

we also know that Ma = 0 and M_{Do} = 0 being extreme hinge supports.

Spans AB and BC

When we consider these spans and compare them with the derivation, we find that situation is similar so both ha and hc terms are positive and equal to 12 mm using three-moment equation.

$$Ma\left(\frac{6}{2Ic}\right) + 2Mb\left(\frac{6}{2Ic} + \frac{8}{4Ic}\right) + Mc\left(\frac{8}{4Ic}\right) = \frac{6E \times 12 \times 10^{-3}}{6} + \frac{6E \times 12 \times 10^{-3}}{8}$$

put Ma = 0, simplify and multiply by Ic

$$2Mb\,(3 + 2) + Mc\,(2) = EI_C \times 12 \times 10^{-3} + 0.75\,EI_C \times 12 \times 10^{-3}$$

put EI = 800

$10\,Mb + 2\,Mc = 9.6 + 7.2 = 16.8$ divide by 10

$Mb + 0.2\,Mc = 1.68$ (1)

Spans BC and CD

Comparing these two spans with the derivation, we notice that ha term is equal to – 12mm and hc term is zero.

$$Ma\left(\frac{8}{4Ic}\right) + 2Mc\left(\frac{8}{4Ic} + \frac{6}{3Ic}\right) + Md\left(\frac{6}{3Ic}\right) = \frac{6E(-12 \times 10^{-3})}{8} + 0$$

Simplify and multiply by Ic

$2\,Mb + 8\,Mc + 2\,Md = -7.2$ divide by 2

$Mb + 4\,Mc + Md = -3.6$ (2)

Spans CD and DD_O

There is no load and settlement on these two spans so right handside of equation is zero

$$Mc\left(\frac{6}{3Ic}\right) + 2Md\left(\frac{6}{3Ic} + \frac{Lo}{\infty}\right) + Mdo\left(\frac{Lo}{\infty}\right) = 0$$

We know that Mdo = 0; $\frac{Lo}{\infty} = 0$

Simplify and multiply by Ic

$2\,Mc + 4\,Md = 0$ divide by 2

$Mc + 2\,Md = 0$ (3)

Above three linear simultaneous equations which are solved. Subtract (2) from (1)

$$\begin{array}{ll} Mb + 0.2\,Mc & = 1.68 \\ Mb + 4\,Mc + Md & = -3.6 \\ \hline -3.8\,Mc - Md & = 5.26 \end{array} \quad (4)$$

Now multiply equation (4) by 2 and add to equation (3)

$$\begin{array}{ll} -7.6\,Mc - 2\,Md & = 10.56 \\ Mc + 2\,Md & = 0 \\ \hline -6.6\,Mc & = 10.56 \end{array}$$

Mc = -1.6 KN-m

$$Md = -\frac{Mc}{2} = +0.8$$

Mb = 2 KN-m

Plot end moment diagram. Add and subtract equal areas on spans BC and CD and apply conjugate beam method.

$$A1 = \frac{1}{2} \times 6 \times 2 = 6$$

$$A2 = \frac{1}{2} \times 8 \times 2 = 8$$

$$A3 = \frac{1}{2} \times 6 \times 0.8 = 2.4$$

$$A4 = \frac{1}{2} \times 8 \times 1.6 = 6.4$$

$$A5 = \frac{1}{2} \times 6 \times 1.6 = 4.8$$

Compute slopes at supports.

θa = Slope due to settlement (configuration) + due to end moments

$$= \frac{12 \times 10^{-3}}{6} + \frac{1}{2EIc}\left[\frac{A1}{3}\right] = \frac{12 \times 10^{-3}}{6} + \frac{1}{1600}\left[\frac{6}{3}\right] = 3.25 \times 10^{-3} \text{ rad.}$$

Span AB

$$\theta b = \frac{12 \times 10^{-3}}{6} + \frac{1}{2EIc}\left[-\frac{2}{3}A1\right] = \frac{12 \times 10^{-6}}{6} + \frac{1}{1600}\left[-\frac{2}{3} \times 6\right]$$

$$= -5 \times 10^{-4} \text{ rad.}$$

Span BC

$$\theta b = \frac{12 \times 10^{-3}}{8} + \frac{1}{4EIc}\left[\frac{2}{3}A2 - \frac{1}{3}A4\right] = \frac{12 \times 10^{-3}}{8} + \frac{1}{4 \times 800}\left[\frac{2}{3} \times 8 - \frac{1}{3} \times 6.4\right]$$

$$\theta b = -5 \times 10^{-4} \text{ rad.}$$

$$\theta c = \frac{12 \times 10^{-3}}{8} + \frac{1}{4EIc}\left[-\frac{1}{3}A2 + \frac{2}{3}A4\right]$$

$$\theta c = -1 \times 10^{-3} \text{ rad.}$$

Span CD

$$\theta c = 0 + \frac{1}{3EIc}\left[\frac{1}{3}A3 - \frac{2}{3}A5\right] = \frac{1}{3 \times 800}\left[\frac{1}{3} \times 2.4 - \frac{2}{3} \times 4.8\right]$$

$$\theta c = -1 \times 10^{-3} \text{ rad.}$$

$$\theta d = 0 + \frac{1}{3EIc}\left[-\frac{2}{3}A3 + \frac{1}{3}A5\right] = 0 + \frac{1}{3 \times 800}\left[-\frac{2}{3} \times 2.4 + \frac{1}{3} \times 4.8\right]$$

$\theta d = 0$ (Fixed end)

Checks on slopes have been satisfied so computed moment values are correct. Now beam is determinate. SFD and BMD can be plotted.

Resolve same problem, for a differential sinking of 12 mm at support C. we get the following equations.

$$Mb + 0.2\,Mc = -0.72 \quad (1)$$
$$Mb + 4\,Mc + Md = 8.4 \quad (2)$$
$$Mc + 2\,Md = -4.8 \quad (3)$$

Solution gives

$Mc = +3.49$

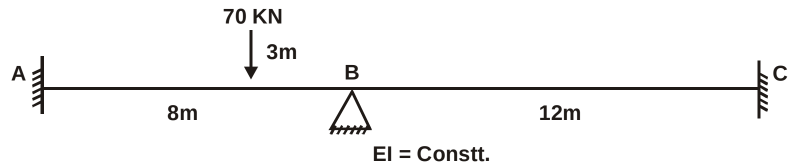
$Md = -4.145$

$Mb = -1.418$

apply continuity checks and plot SFD and BMD.

Unsolved Examples:

Solve the following loaded beams by three-moment equations.

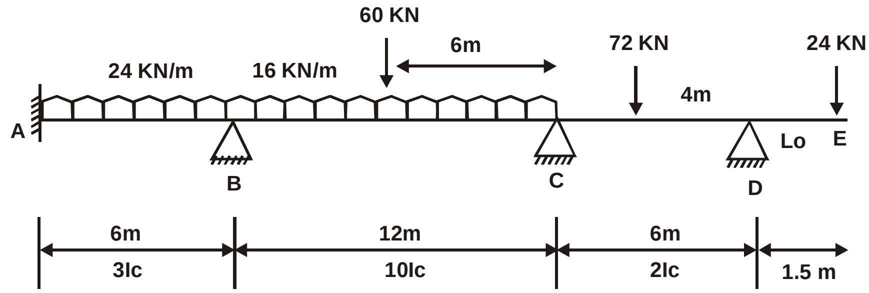


Final equations:

Ma + 0.5 Mb	= – 90.312	(1)
Ma + 5 Mb + 1.5 Mc	= – 213.12	(2)
Mb + 2 Mc	= 0	(3)

End Moment Values:

Mc = 16.41
Mb = – 32.82
Ma = – 73.91

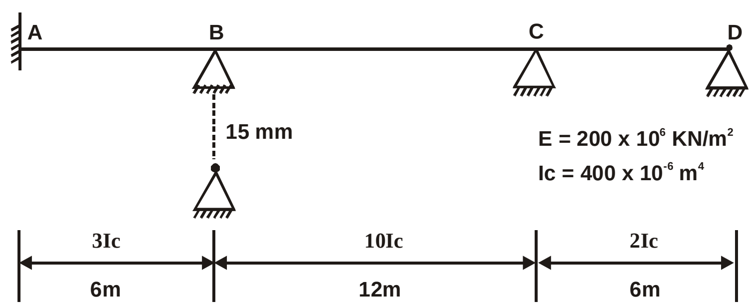


Final Equations:

2 Ma + Mb	= – 216	(1)
2 Ma + 6.4 Mb + 1.2 Mc	= – 1555.2	(2)
1.2 Mb + 8.4 Mc	= – 1495.2	(3)

End moment values:

Ma = – 0.361 KN-m
Mb = – 215.28 Kn-m
Mc = – 147.25 Kn-m



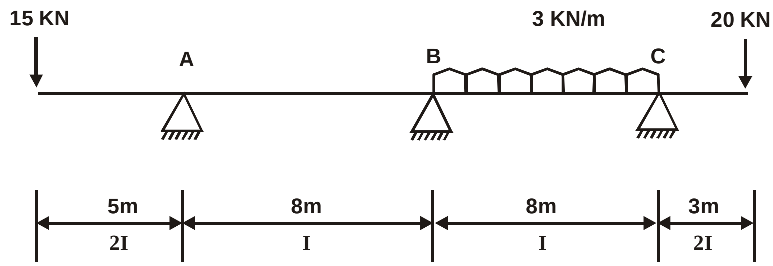
Final Equations:

2 Ma + Mb	= – 600	(1)
2 Ma + 6.4 Mb + 1.2 Mc	= 1800	(2)

1.2 Mb + 8.4 Mc = – 600 (3)

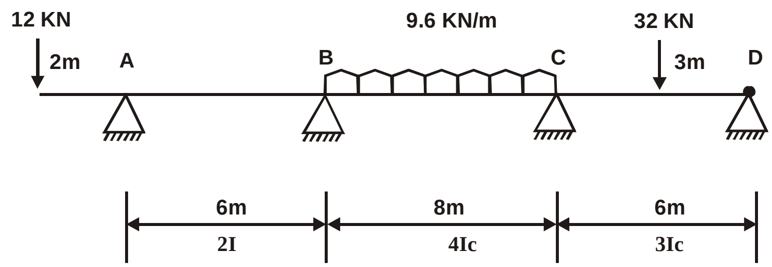
End moment values:

Ma = – 537.69 KN-m
Mb = 475.38
Mc = – 139.34 KN-m



End moment values:

Ma = – 75 KN-m
Mb = 21.75
Mc = – 60 KN-m

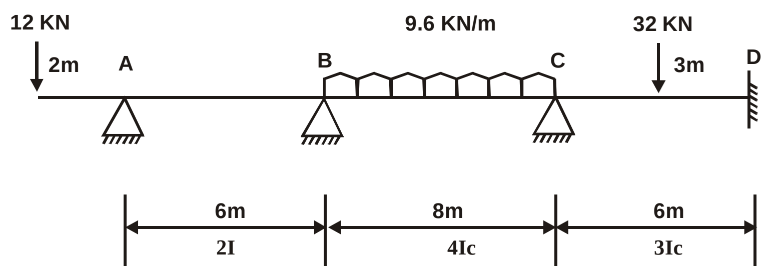


Final equations:

10 Mb + 2 Mc = – 235.2 (1)
2 Mb + 8 Mc = – 451.2 (2)

End moment values:

Ma = – 24 KN-m
Mb = – 12.88
Mc = – 53.18
Md = 0



Final equations:

$10\ M_b + 2\ M_c = -235.2$ (1)

$2\ M_b + 8\ M_c + 2\ M_D = -451.2$ (2)

$2\ M_c + 4\ M_D = -144$ (3)

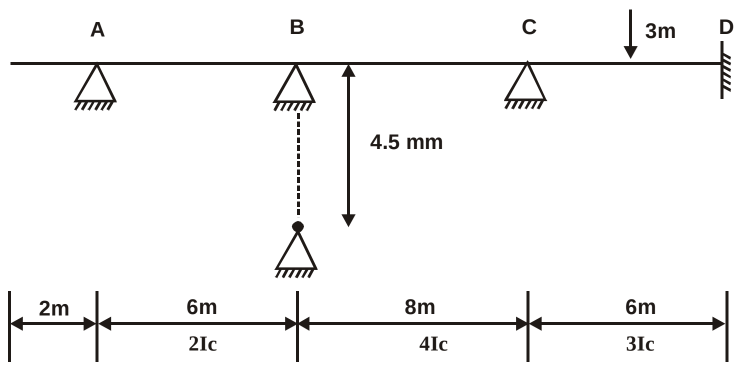
End moment values:

$M_a = -24$ KN-m

$M_b = -13.455$

$M_c = -50.33$

$M_d = -10.835$



Final equations:

$10\ M_b + 2\ M_c = 6.3$ (1)

$2\ M_b + 8\ M_c + 2\ M_d = -2.7$ (2)

$2\ M_c + 2\ M_D = 0$ (3)

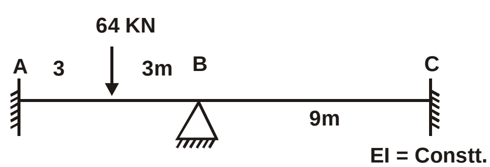
End moment values:

$M_a = 0$

$M_b = 0.7714$

$M_c = -0.707$

$M_d = 0.707$



Final equations:

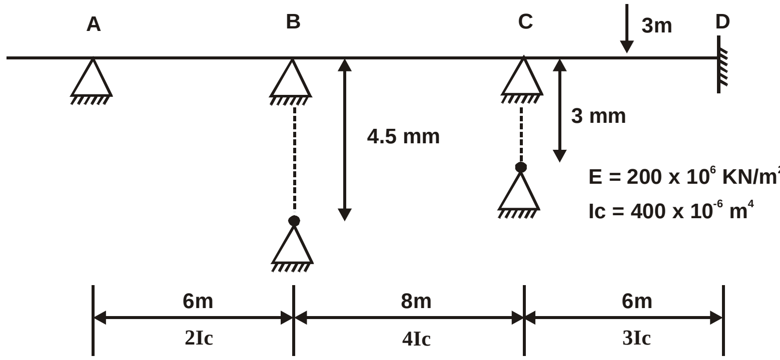
$2\ M_a + M_b = -144$ (1)

$2\ M_a + 10\ M_b + 3\ M_c = -288$ (2)

$M_b + 2\ M_c = 0$ (3)

End moment values:

Mb = -19.2
Mc = 9.6
Ma = – 62.4



Final equations:

Mb + 0.2 Mc	= 5.4	(1)
Mb + 4 Mc + MD	= – 1.5	(2)
Mc + 2 MD	= – 12	(3)

End moment values:
Ma = 0
Mb = 5.45
Mc = – 0.27
MD = – 5.86