Analysis of Statically Indeterminate Structures by the Displacement Method

Methods of structural analysis

- The methods are classified into two groups:

1. <u>Force method of analysis</u> (Statics of building structures I)
   - Primary unknowns are forces and compatibility of displacements is written in terms of pre-selected redundant reactions and flexibility coefficients using force displacement relations.
   - The unknown redundant reactions are evaluated solving these equations.
   - The remaining reactions are obtained from equations of equilibrium.

Methods of structural analysis

- The methods are classified into two groups:

  2. <u>Displacement method of analysis</u>
     - Primary unknowns are displacements.
     - Equilibrium equations are written by expressing the unknown joint displacements in terms of loads by using load-displacement relations.
     - Unknown joint displacements are calculated by solving equilibrium equations.
     - In the next step, the unknown reactions are computed from compatibility equations using force displacement relations.

Displacement method

- This method follows essentially the same steps for both statically determinate and indeterminate structures.
- Once the structural model is defined, the unknowns (joint rotations and translations) are automatically chosen unlike the force method of analysis (hence, this method is preferred to computer implementation).

Displacement method

1. Slope-Deflection Method
   - In this method it is assumed that all deformations are due to bending only. Deformations due to axial forces are neglected.
2. Direct Stiffness Method
   - Deformations due to axial forces are not neglected.

The Slope-deflection method was used for many years before the computer era. After the revolution occurred in the field of computing direct stiffness method is preferred.

Slope-Deflection Method: Beams

Slope-Deflection Method: Beams

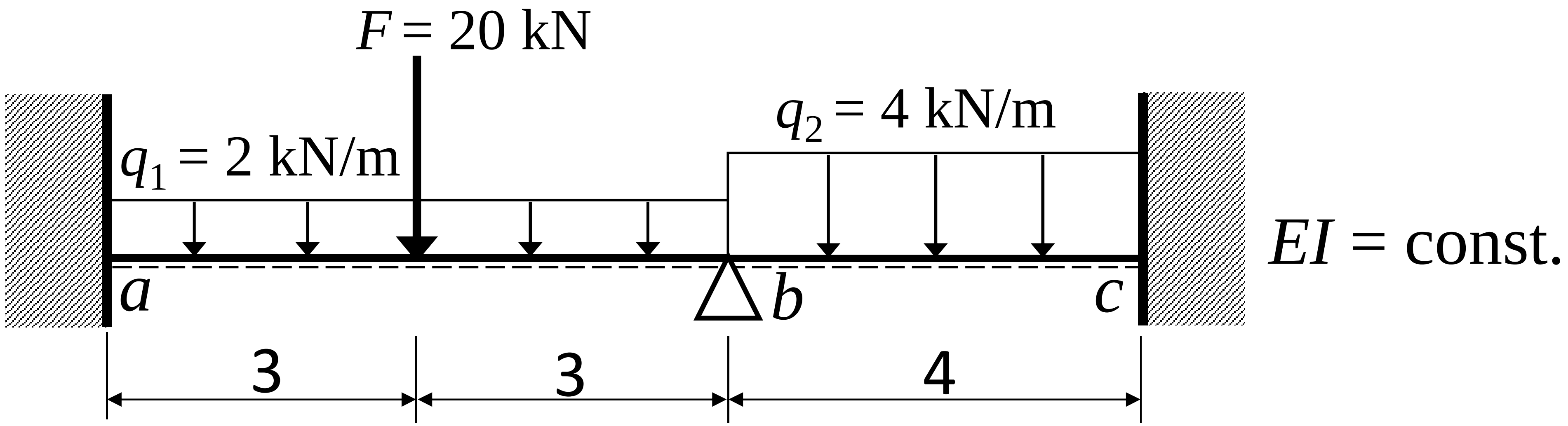
- Application of Slope-Deflection Equations to <u>Statically Indeterminate Beams:</u>
  - The procedure is the same whether it is applied to beams or frames.
  - It may be summarized as follows:

1. Identify all kinematic degrees of freedom for the given problem. Degrees of freedom are treated as unknowns in slope-deflection method.
2. Determine the fixed end moments at each end of the span to applied load (using table).
3. Express all internal end moments by slope-deflection equations in terms of:
   - fixed end moments
   - near end and far end joint rotations

Slope-Deflection Method: Beams

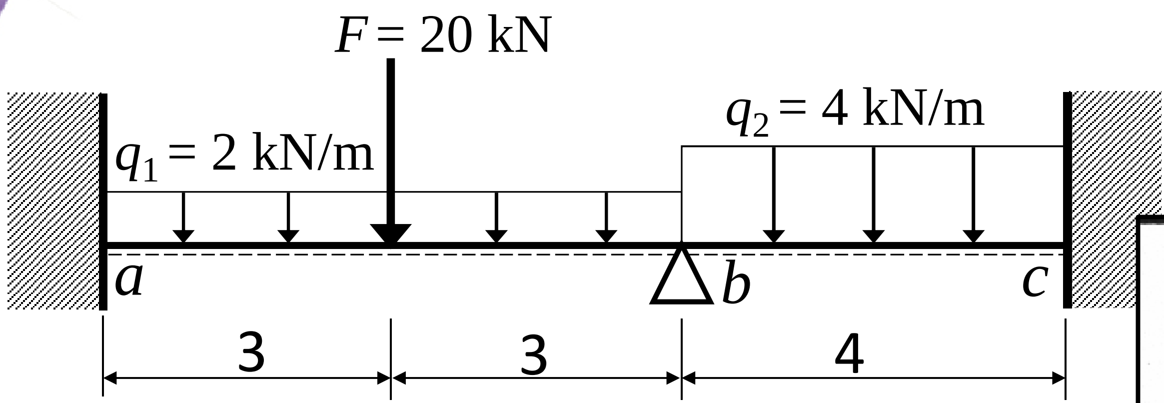
4. Write down one equilibrium equation for each unknown joint rotation. Write down as many equilibrium equations as there are unknown joint rotations. Solve the set of equilibrium equations for joint rotations.
5. Now substituting these joint rotations in the slope-deflection equations evaluate the end moments.
6. Evaluate shear forces and reactions.
7. Draw bending moment and shear force diagrams.

Slope-Deflection Method: Beams
Example 1



1. Degrees of freedom
   - The continuous beam is kinematically indeterminate to first degree. Only one joint rotation φ_b is unknown.

Slope-Deflection Method: Beams
Example 1



2. Fixed end moments are calculated referring to the table.

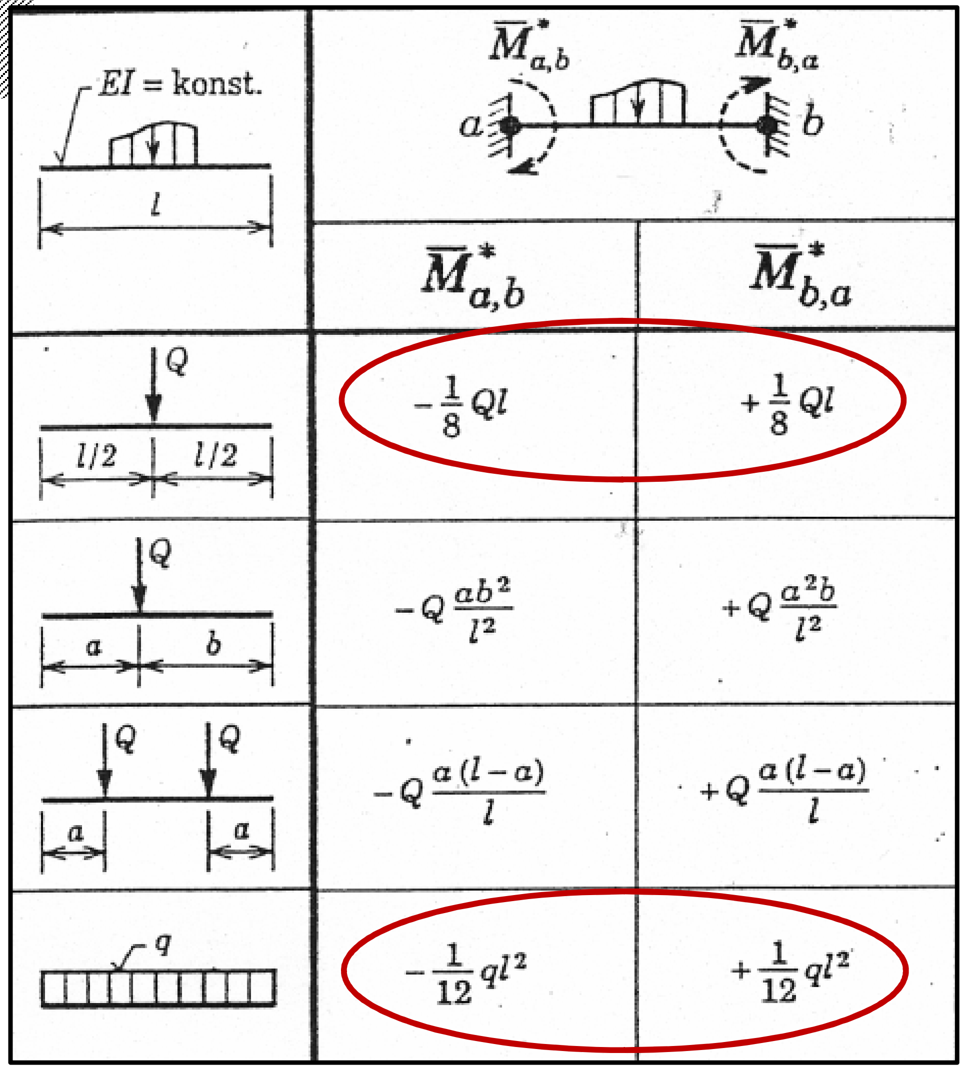
$$\overline{M}_{ab} = -\frac{1}{8} \cdot F \cdot l_{ab} - \frac{1}{12} \cdot q_1 \cdot l_{ab}^2$$

$$\overline{M}_{ab} = -\frac{1}{8} \cdot 20 \cdot 6 - \frac{1}{12} \cdot 2 \cdot 6^2 = -21\,\text{kNm}$$

$$\overline{M}_{ba} = +\frac{1}{8} \cdot F \cdot l_{ab} + \frac{1}{12} \cdot q_1 \cdot l_{ab}^2 = 21\,\text{kNm}$$

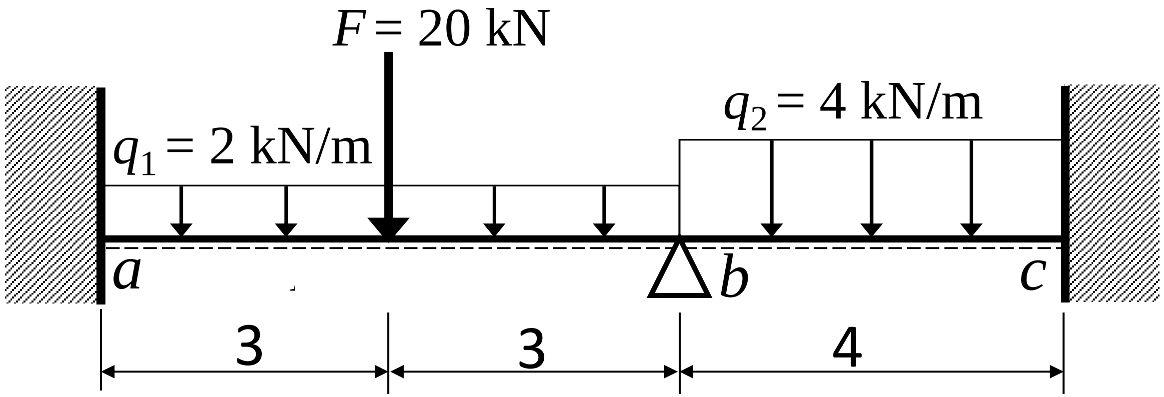
$$\overline{M}_{bc} = -\frac{1}{12} \cdot q_2 \cdot l_{bc}^2 = -\frac{1}{12} \cdot 4 \cdot 4^2 = -5{,}33\,\text{kNm}$$

$$\overline{M}_{cb} = +\frac{1}{12} \cdot q_2 \cdot l_{bc}^2 = 5{,}33\,\text{kNm}$$



EI = konst., l	$\overline{M}^*_{a,b}$	$\overline{M}^*_{b,a}$
Q; $l/2$, $l/2$	$-\frac{1}{8}Ql$	$+\frac{1}{8}Ql$
Q; a, b	$-Q\frac{ab^2}{l^2}$	$+Q\frac{a^2b}{l^2}$
Q, Q; a, a	$-Q\frac{a(l-a)}{l}$	$+Q\frac{a(l-a)}{l}$
q	$-\frac{1}{12}ql^2$	$+\frac{1}{12}ql^2$

Slope-Deflection Method: Beams
Example 1



3. Express internal end moments by slope-deflection equations.

$$M_{ab} = \boxed{\overline{M}_{ab}} + \frac{2\boxed{EI}}{l}\left(2\cdot\boxed{\varphi_a} + \boxed{\varphi_b}\right)$$

far end joint rotation

fixed end moment

flexural rigidity

near end joint rotation

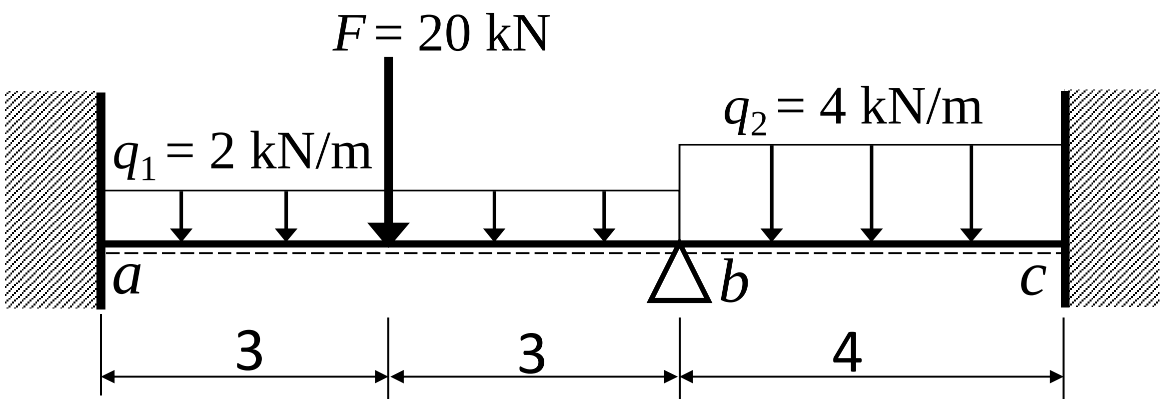
$$M_{ab} = \overline{M}_{ab} + \frac{2EI}{l_{ab}}\left(2\cdot\varphi_a + \varphi_b\right) = -21 + \frac{2\,EI}{6}\left(2\cdot 0 + \varphi_b\right) = -21 + \frac{EI}{3}\varphi_b$$

$$M_{ba} = \overline{M}_{ba} + \frac{2EI}{l_{ab}}\left(2\cdot\varphi_b + \varphi_a\right) = 21 + \frac{2\,EI}{6}\left(2\cdot\varphi_b + 0\right) = 21 + \frac{2\cdot EI}{3}\varphi_b$$

$$M_{bc} = \overline{M}_{bc} + \frac{2EI}{l_{bc}}\left(2\cdot\varphi_b + \varphi_c\right) = -5{,}33 + \frac{2\,EI}{4}\left(2\cdot\varphi_b + 0\right) = -5{,}33 + EI\cdot\varphi_b$$

$$M_{cb} = \overline{M}_{cb} + \frac{2EI}{l_{bc}}\left(2\cdot\varphi_c + \varphi_b\right) = 5{,}33 + \frac{2\,EI}{4}\left(2\cdot 0 + \varphi_b\right) = 5{,}33 + \frac{EI}{2}\cdot\varphi_b$$

Slope-Deflection Method: Beams
Example 1



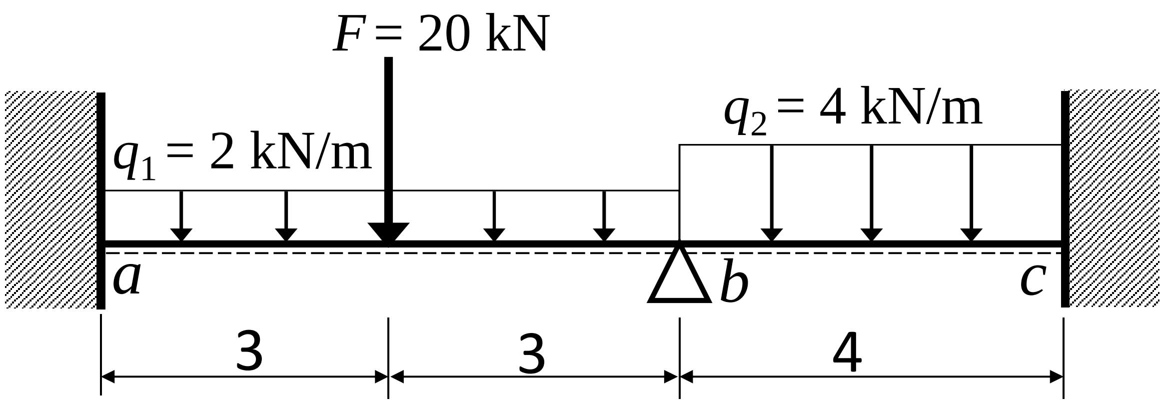
4. Equilibrium equations (write one equilibrium equation for each unknown joint rotation)
   - End moments are expressed in terms of unknown rotation φ_b. Now, the required equation to solve for the rotation φ_b is the moment equilibrium equation at support b.

$$\sum M_b = 0: \quad M_{ba} + M_{bc} = 0$$

$$21 + \frac{2EI}{3} \cdot \varphi_b - 5{,}33 + EI \cdot \varphi_b = 0$$

$$1{,}667 \cdot EI \cdot \varphi_b + 15{,}667 = 0 \Rightarrow \varphi_b = -\frac{15{,}667}{1{,}667 \cdot EI} = -\frac{9{,}4}{EI}$$

Slope-Deflection Method: Beams
Example 1



5. End moments

- After evaluating φ_b, substitute it to evaluate beam end moments.

$$M_{ab} = -21 + \frac{EI}{3}\varphi_b = -21 + \frac{EI}{3}\cdot\left(-\frac{9,4}{EI}\right) = -24,13\ \text{kNm}$$

$$M_{ba} = 21 + \frac{2\cdot EI}{3}\varphi_b = 21 + \frac{2\cdot EI}{3}\cdot\left(-\frac{9,4}{EI}\right) = 14,73\ \text{kNm}$$

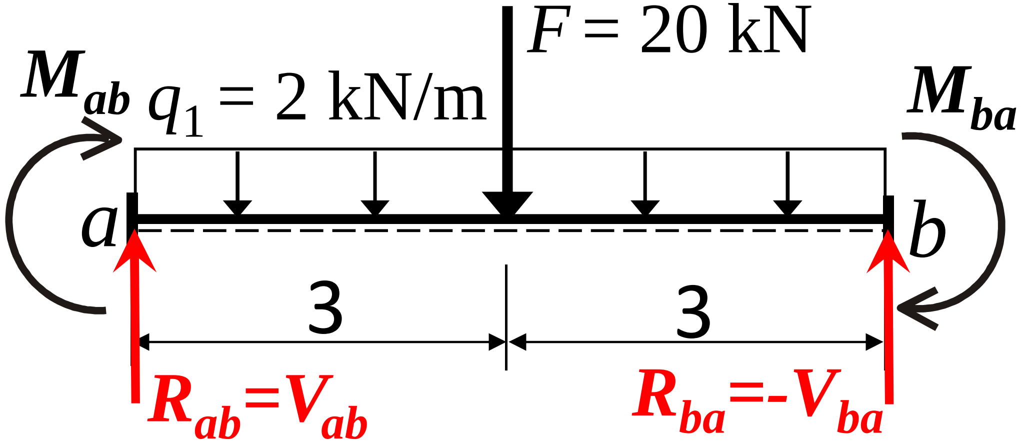
$$M_{bc} = -5,33 + EI\cdot\varphi_b = -5,33 + EI\cdot\left(-\frac{9,4}{EI}\right) = -14,73\ \text{kNm}$$

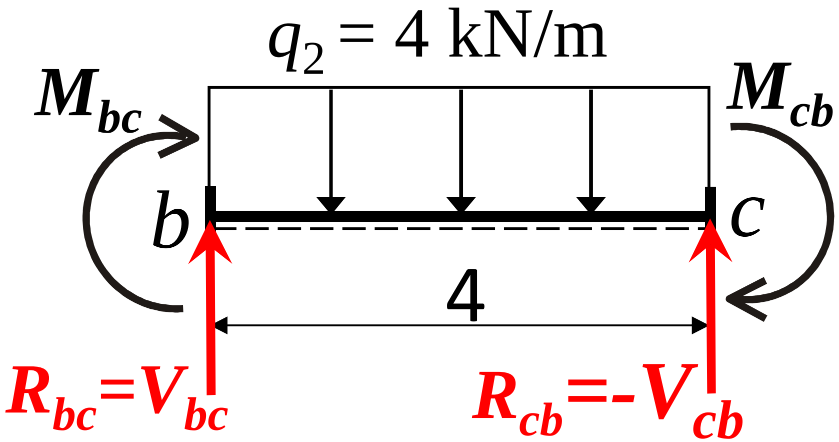
$$M_{cb} = 5,33 + \frac{EI}{2}\cdot\varphi_b = 5,33 + \frac{EI}{2}\cdot\left(-\frac{9,4}{EI}\right) = 0,63\ \text{kNm}$$

Slope-Deflection Method: Beams Example 1

6. Shear forces and reactions

- Now, reactions at supports are evaluated using equilibrium equations. Shear forces are equal to plus/minus this reactions.





$$V_{ab} = R_{ab} = \frac{1}{l_{ab}}\left(F\cdot\frac{l_{ab}}{2} + q_1\cdot\frac{l_{ab}^2}{2} - M_{ab} - M_{ba}\right)$$

$$V_{ab} = \frac{1}{6}\left(20\cdot 3 + 2\cdot\frac{6^2}{2} + 24{,}13 - 14{,}73\right) = 17{,}57\text{ kN}$$

$$V_{ba} = -R_{ba} = -\frac{1}{l_{ab}}\left(F\cdot\frac{l_{ab}}{2} + q_1\cdot\frac{l_{ab}^2}{2} + M_{ab} + M_{ba}\right)$$

$$V_{ba} = -\frac{1}{6}\left(20\cdot 3 + 2\cdot\frac{6^2}{2} - 24{,}13 + 14{,}73\right) = -14{,}43\text{ kN}$$

$$V_{bc} = R_{bc} = \frac{1}{l_{bc}}\left(q_2\cdot\frac{l_{cb}^2}{2} - M_{bc} - M_{cb}\right)$$

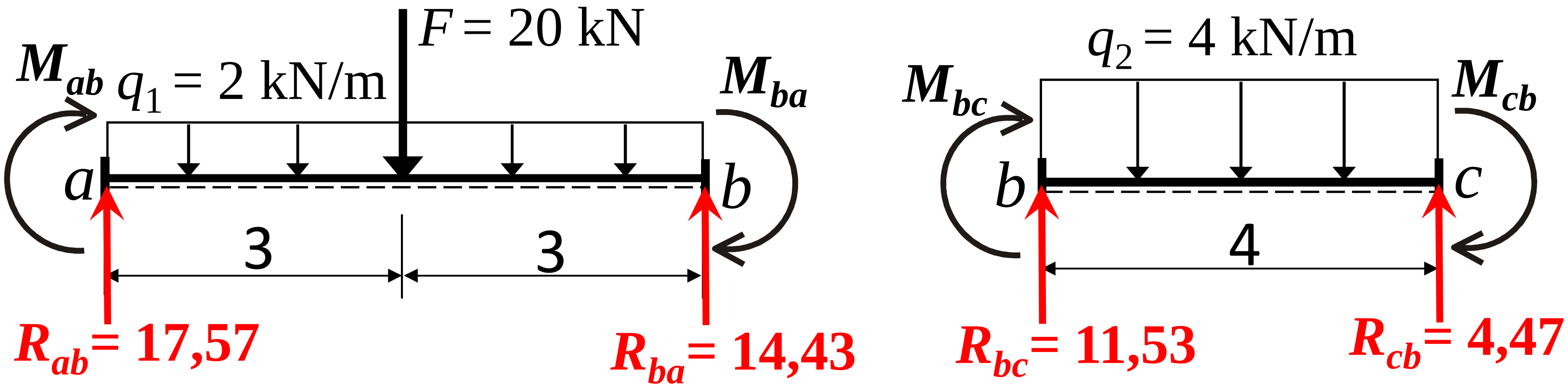
$$V_{bc} = \frac{1}{4}\left(4\cdot\frac{4^2}{2} + 14{,}73 - 0{,}63\right) = 11{,}53\text{ kN}$$

$$V_{cb} = -R_{cb} = -\frac{1}{l_{bc}}\left(q_2\cdot\frac{l_{cb}^2}{2} + M_{bc} + M_{cb}\right)$$

$$V_{cb} = -\frac{1}{4}\left(4\cdot\frac{4^2}{2} - 14{,}73 + 0{,}63\right) = -4{,}47\text{ kN}$$

Slope-Deflection Method: Beams Example 1

6. Shear forces and reactions
   - Now, reactions at supports are evaluated using equilibrium equations. Shear forces are equal to plus/minus this reactions.



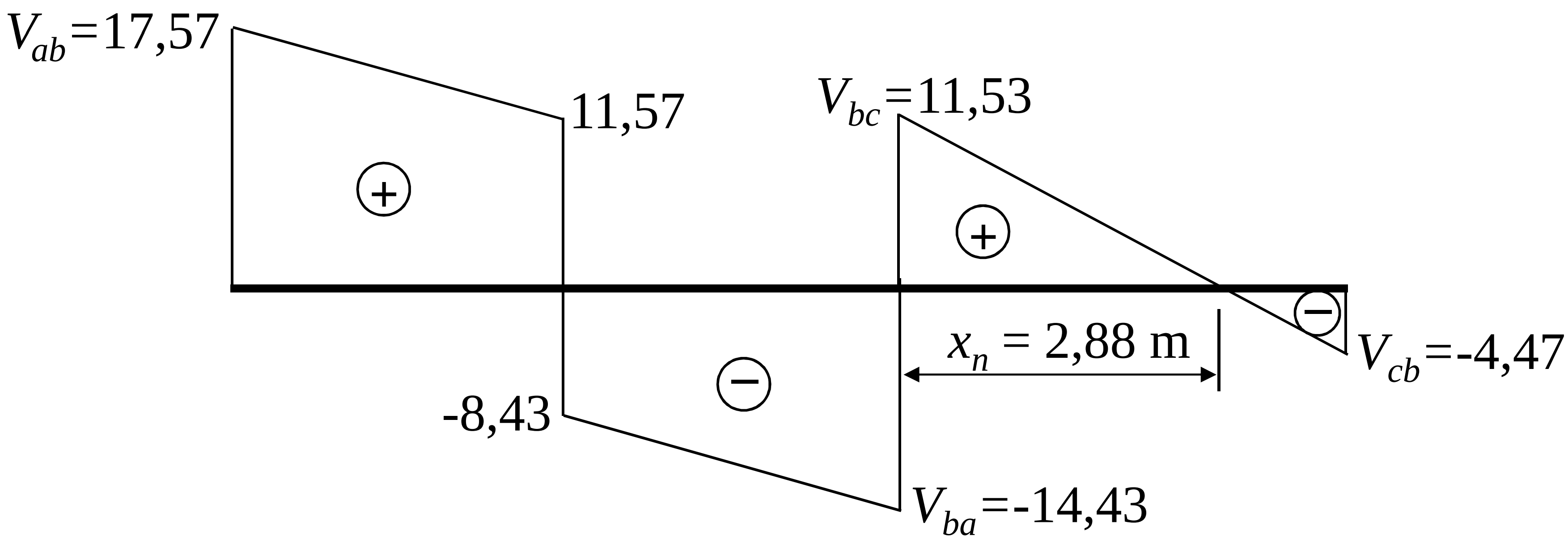
$$R_{az} = R_{ab} = 17{,}57\,\text{kN}$$

$$R_{bz} = R_{ba} + R_{bc} = 14{,}43 + 11{,}53 = 25{,}96\,\text{kN}$$

$$R_{cz} = R_{cb} = 4{,}47\,\text{kN}$$

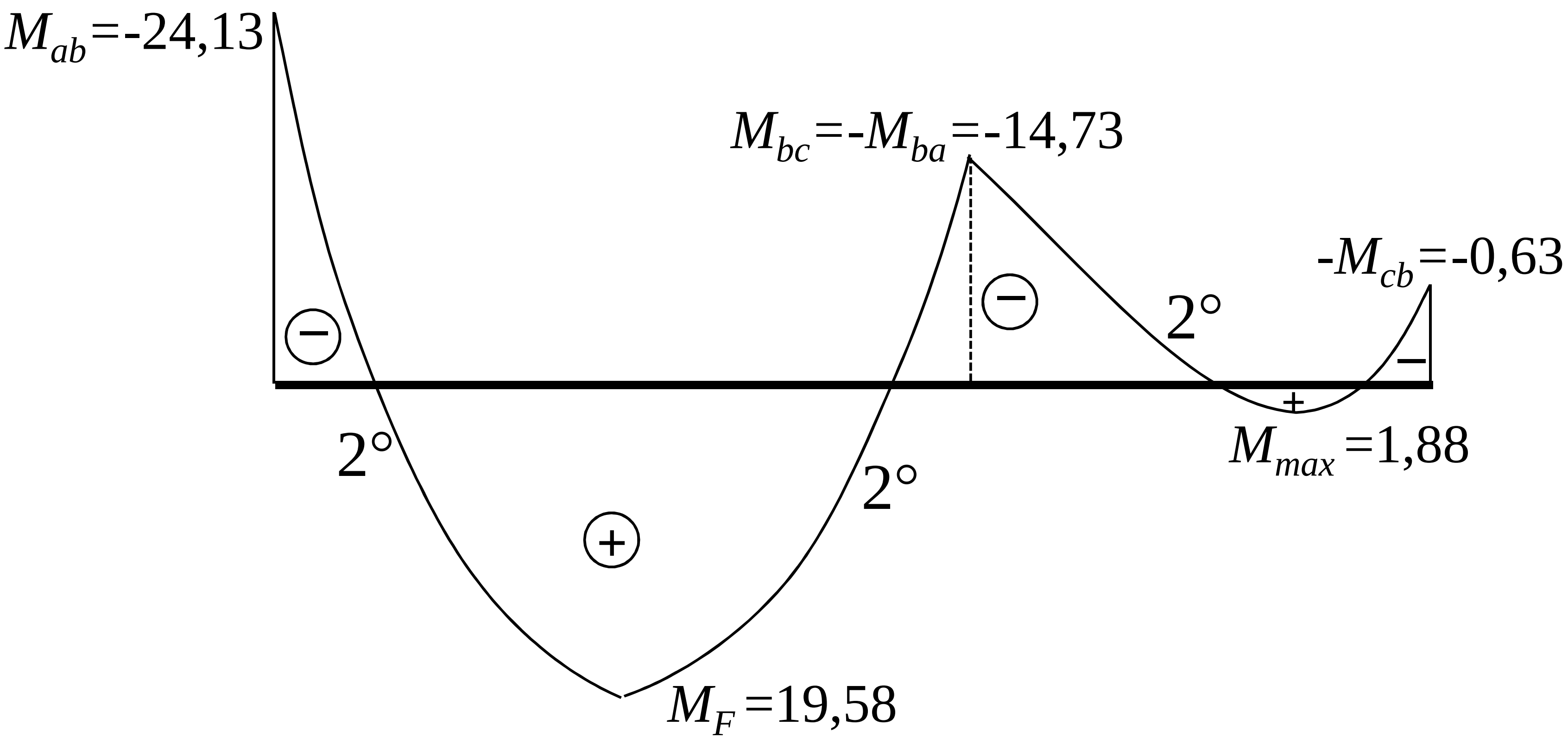
Slope-Deflection Method: Beams
Example 1

7. Draw <u>shear force</u> and bending moment diagrams.



Slope-Deflection Method: Beams
Example 1

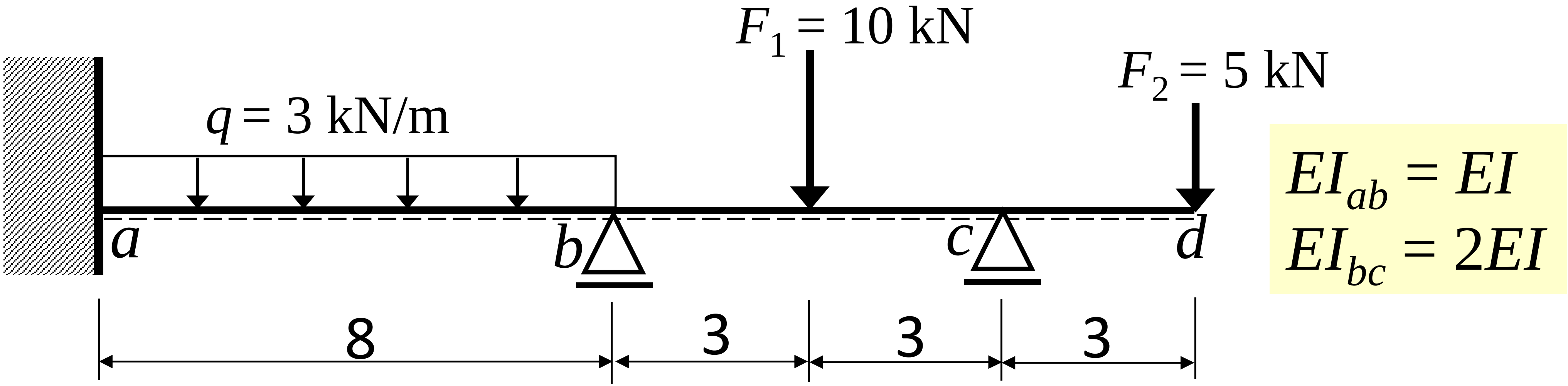
7. Draw shear force and <u>bending moment</u> diagrams.



$$M_F = M_{ab} + R_{ab} \cdot \frac{l_{ab}}{2} - q_1 \cdot \left(\frac{l_{ab}}{2}\right)^2 \cdot \frac{1}{2} = -24{,}13 + 17{,}57 \cdot 3 - 2 \cdot \frac{3^2}{2} = 19{,}58 \text{ kNm}$$

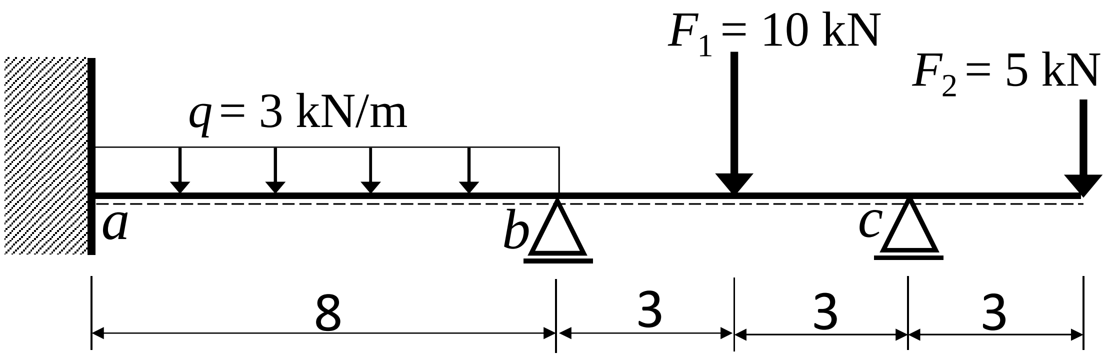
$$M_{max} = M_{bc} + R_{bc} \cdot x_n - q_2 \cdot \frac{x_n^2}{2} = -14{,}73 + 11{,}53 \cdot 2{,}88 - 4 \cdot \frac{2{,}88^2}{2} = 1{,}88 \text{ kNm}$$

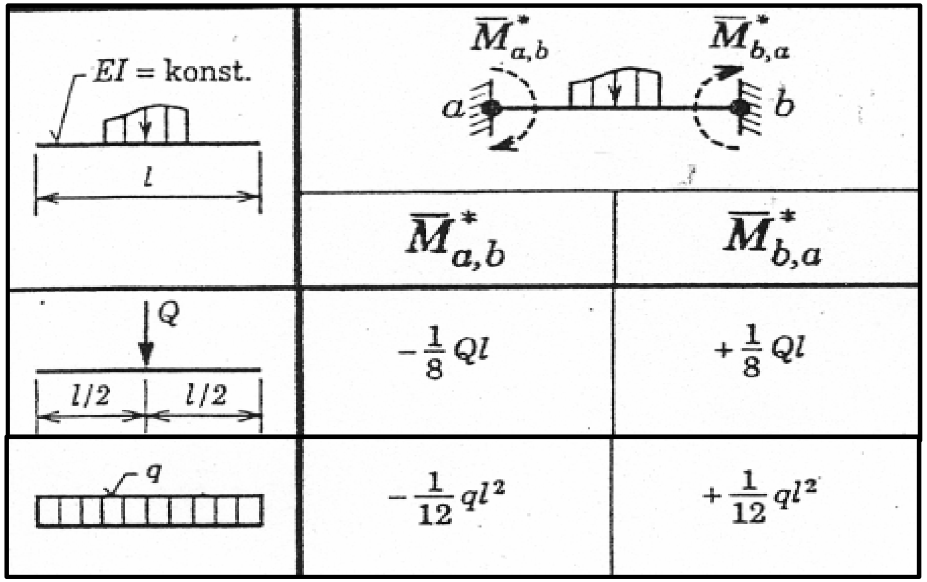
Slope-Deflection Method: Beams
Example 2



1. Degrees of freedom
   - The continuous beam is kinematically indeterminate to second degree.
     - The 1st **possibility of solution** – two unknown joint rotation φ_b, φ_c ($\varphi_a = 0$) - two required equations to solve for the rotation φ_b, φ_c are the moment equilibrium equations at support b and c.

Slope-Deflection Method: Beams
Example 2





	$\overline{M}^*_{a,b}$	$\overline{M}^*_{b,a}$
EI = konst., l	$\overline{M}^*_{a,b}$	$\overline{M}^*_{b,a}$
Q, $l/2$, $l/2$	$-\frac{1}{8}Ql$	$+\frac{1}{8}Ql$
q	$-\frac{1}{12}ql^2$	$+\frac{1}{12}ql^2$

2. Fixed end moments are calculated referring to the table.

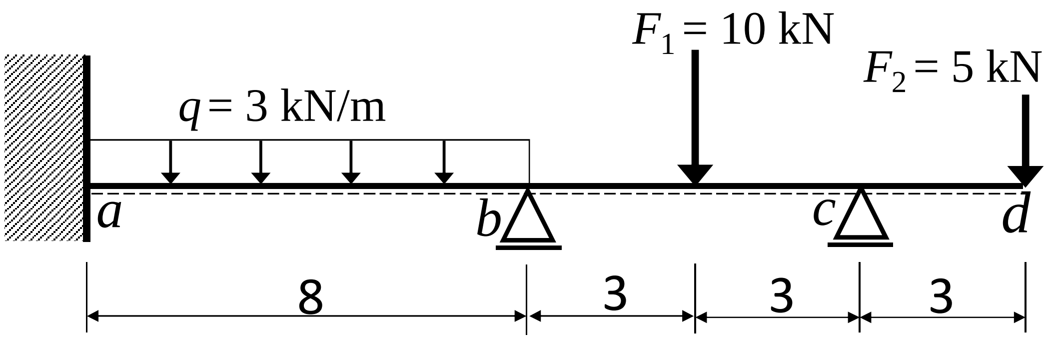
$$\overline{M}_{ab} = -\frac{1}{12}\cdot q\cdot l_{ab}^2 = -\frac{1}{12}\cdot 3\cdot 8^2 = -16\text{ kNm}$$

$$\overline{M}_{ba} = +\frac{1}{12}\cdot q\cdot l_{ab}^2 = 16\text{ kNm}$$

$$\overline{M}_{bc} = -\frac{1}{8}\cdot F\cdot l_{bc} = -\frac{1}{8}\cdot F\cdot l_{bc} = -\frac{1}{8}\cdot 10\cdot 6 = -7{,}5\text{ kNm}$$

$$\overline{M}_{cb} = +\frac{1}{8}\cdot F\cdot l_{bc} = +\frac{1}{8}\cdot F\cdot l_{bc} = +\frac{1}{8}\cdot 10\cdot 6 = 7{,}5\text{ kNm}$$

Slope-Deflection Method: Beams
Example 2



$EI_{ab} = EI$
$EI_{bc} = 2EI$

3. Express internal end moments by slope-deflection equations.

$$M_{ab} = \overline{M}_{ab} + \frac{2EI_{ab}}{l_{ab}}(2\cdot\varphi_a + \varphi_b) = -16 + \frac{2\,EI}{8}(2\cdot 0 + \varphi_b) = -16 + \frac{EI}{4}\varphi_b$$

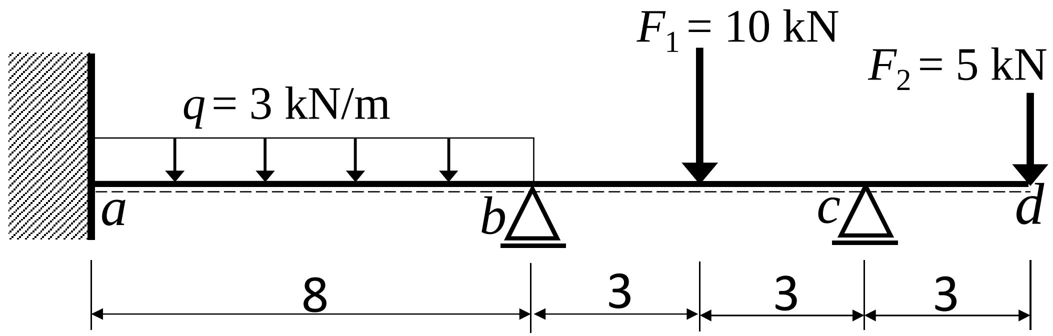
$$M_{ba} = \overline{M}_{ba} + \frac{2EI_{ab}}{l_{ab}}(2\cdot\varphi_b + \varphi_a) = 16 + \frac{2EI}{8}(2\cdot\varphi_b + 0) = 16 + \frac{EI}{2}\varphi_b$$

$$M_{bc} = \overline{M}_{bc} + \frac{2EI_{bc}}{l_{bc}}(2\cdot\varphi_b + \varphi_c) = -7{,}5 + \frac{2\cdot 2EI}{6}(2\cdot\varphi_b + \varphi_c) = -7{,}5 + \frac{4}{3}EI\cdot\varphi_b + \frac{2}{3}EI\cdot\varphi_c$$

$$M_{cb} = \overline{M}_{cb} + \frac{2EI_{bc}}{l_{bc}}(2\cdot\varphi_c + \varphi_b) = +7{,}5 + \frac{2\cdot 2EI}{6}(2\cdot\varphi_c + \varphi_b) = 7{,}5 + \frac{4}{3}EI\cdot\varphi_c + \frac{2}{3}EI\cdot\varphi_b$$

$$M_{cd} = -F_2\cdot 3 = -15\ \text{kNm}$$

Slope-Deflection Method: Beams
Example 2



4. Equilibrium equations (write two equilibrium equations for two unknown joint rotations)
   - End moments are expressed in terms of unknown rotations. Now, the required equations to solve for the rotations are the moment equilibrium equations at supports b and c.

$$\sum M_b = 0: \quad M_{ba} + M_{bc} = 0$$

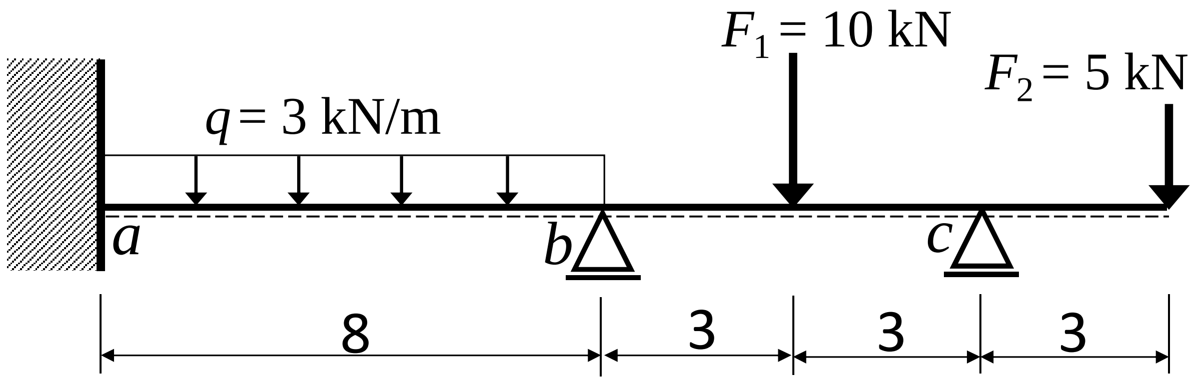
$$\sum M_c = 0: \quad M_{cb} + M_{cd} = 0$$

$$16 + \frac{EI}{2}\varphi_b - 7{,}5 + \frac{4}{3}EI \cdot \varphi_b + \frac{2}{3}EI \cdot \varphi_c = 0$$

$$7{,}5 + \frac{4}{3}EI \cdot \varphi_c + \frac{2}{3}EI \cdot \varphi_b - 15 = 0$$

$$\varphi_b = -\frac{8{,}167}{EI} \qquad \varphi_c = \frac{9{,}708}{EI}$$

Slope-Deflection Method: Beams
Example 2



5. End moments
   - After evaluating φ_b, φ_c, substitute them to evaluate beam end moments.

$$M_{ab} = -16 + \frac{EI}{4}\varphi_b = -16 + \frac{EI}{4}\left(-\frac{8{,}167}{EI}\right) = -18{,}04 \text{ kNm}$$

$$M_{ba} = 16 + \frac{EI}{2}\varphi_b = 16 + \frac{EI}{2}\left(-\frac{8{,}167}{EI}\right) = 11{,}92 \text{ kNm}$$

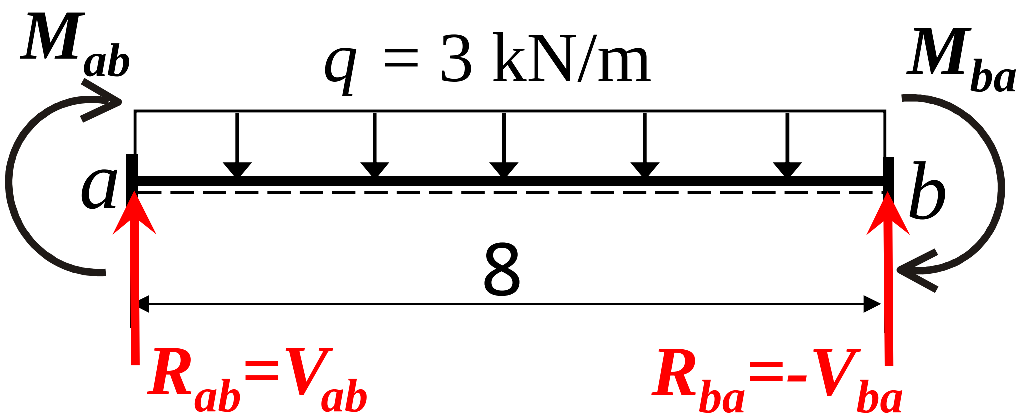
$$M_{bc} = -7{,}5 + \frac{4}{3}EI \cdot \varphi_b + \frac{2}{3}EI \cdot \varphi_c = -7{,}5 + \frac{4}{3}EI \cdot \left(-\frac{8{,}167}{EI}\right) + \frac{2}{3}EI \cdot \frac{9{,}708}{EI} = -11{,}92 \text{ kNm}$$

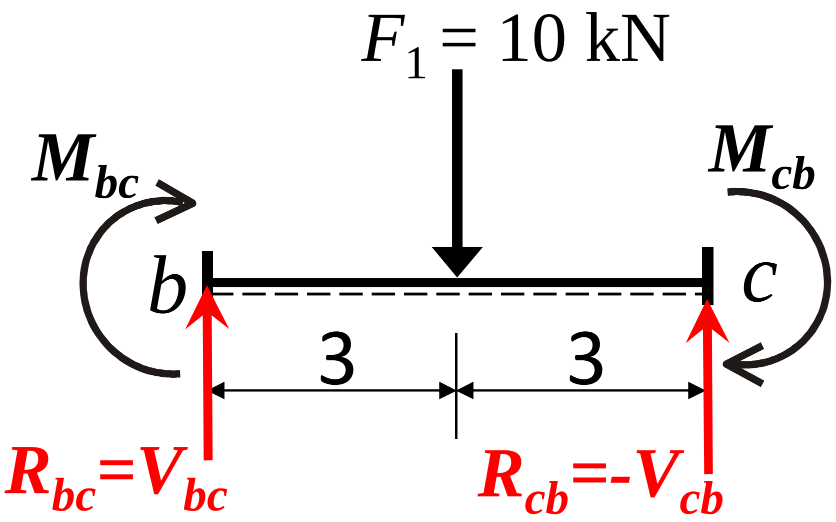
$$M_{cb} = 7{,}5 + \frac{4}{3}EI \cdot \varphi_c + \frac{2}{3}EI \cdot \varphi_b = 7{,}5 + \frac{4}{3}EI \cdot \frac{9{,}708}{EI} + \frac{2}{3}EI \cdot \left(-\frac{8{,}167}{EI}\right) = 15 \text{ kNm}$$

$$M_{cd} = -F_2 \cdot 3 = -15 \text{ kNm}$$

Slope-Deflection Method: Beams Example 2

6. Shear forces and reactions.





$$V_{ab} = R_{ab} = \frac{1}{l_{ab}}\left(q_1 \cdot \frac{l_{ab}^2}{2} - M_{ab} - M_{ba}\right)$$

$$V_{ab} = \frac{1}{8}\left(3 \cdot \frac{8^2}{2} + 18{,}04 - 11{,}92\right) = 12{,}77 \text{ kN}$$

$$V_{ba} = -R_{ba} = -\frac{1}{l_{ab}}\left(q_1 \cdot \frac{l_{ab}^2}{2} + M_{ab} + M_{ba}\right)$$

$$V_{ba} = -\frac{1}{8}\left(3 \cdot \frac{8^2}{2} - 18{,}04 + 11{,}92\right) = -11{,}23 \text{ kN}$$

$$V_{bc} = R_{bc} = \frac{1}{l_{bc}}\left(F_1 \cdot \frac{l_{bc}}{2} - M_{bc} - M_{cb}\right)$$

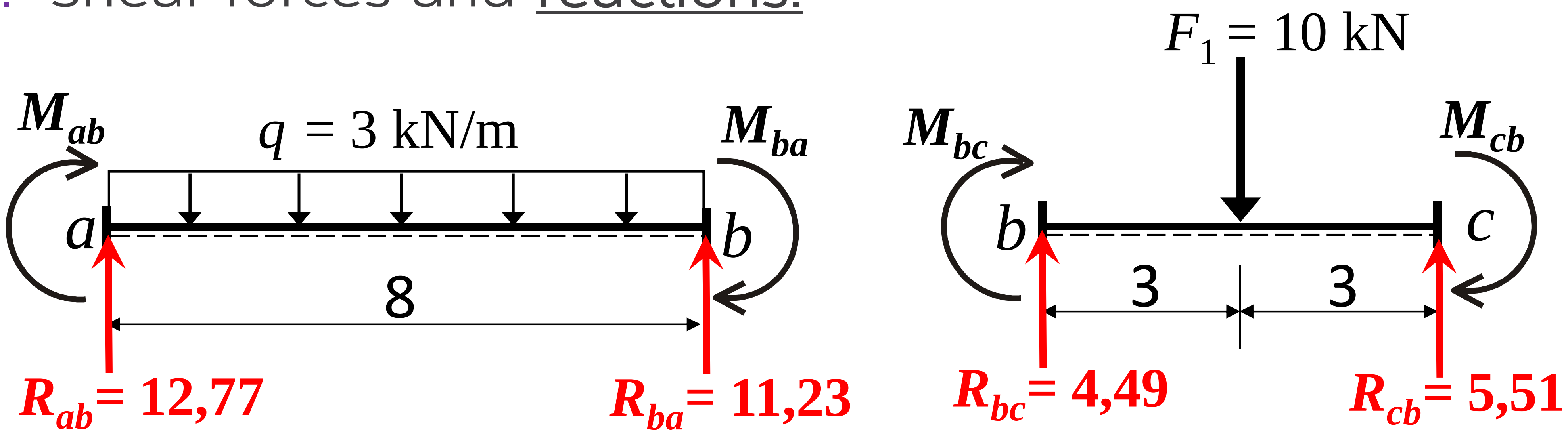
$$V_{bc} = \frac{1}{6}(10 \cdot 3 + 11{,}92 - 15) = 4{,}49 \text{ kN}$$

$$V_{cb} = -R_{cb} = -\frac{1}{l_{bc}}\left(F_1 \cdot \frac{l_{bc}}{2} + M_{bc} + M_{cb}\right)$$

$$V_{cb} = -\frac{1}{6}(10 \cdot 3 - 11{,}92 + 15) = -5{,}51 \text{ kN}$$

Slope-Deflection Method: Beams
Example 2

6. Shear forces and <u>reactions.</u>



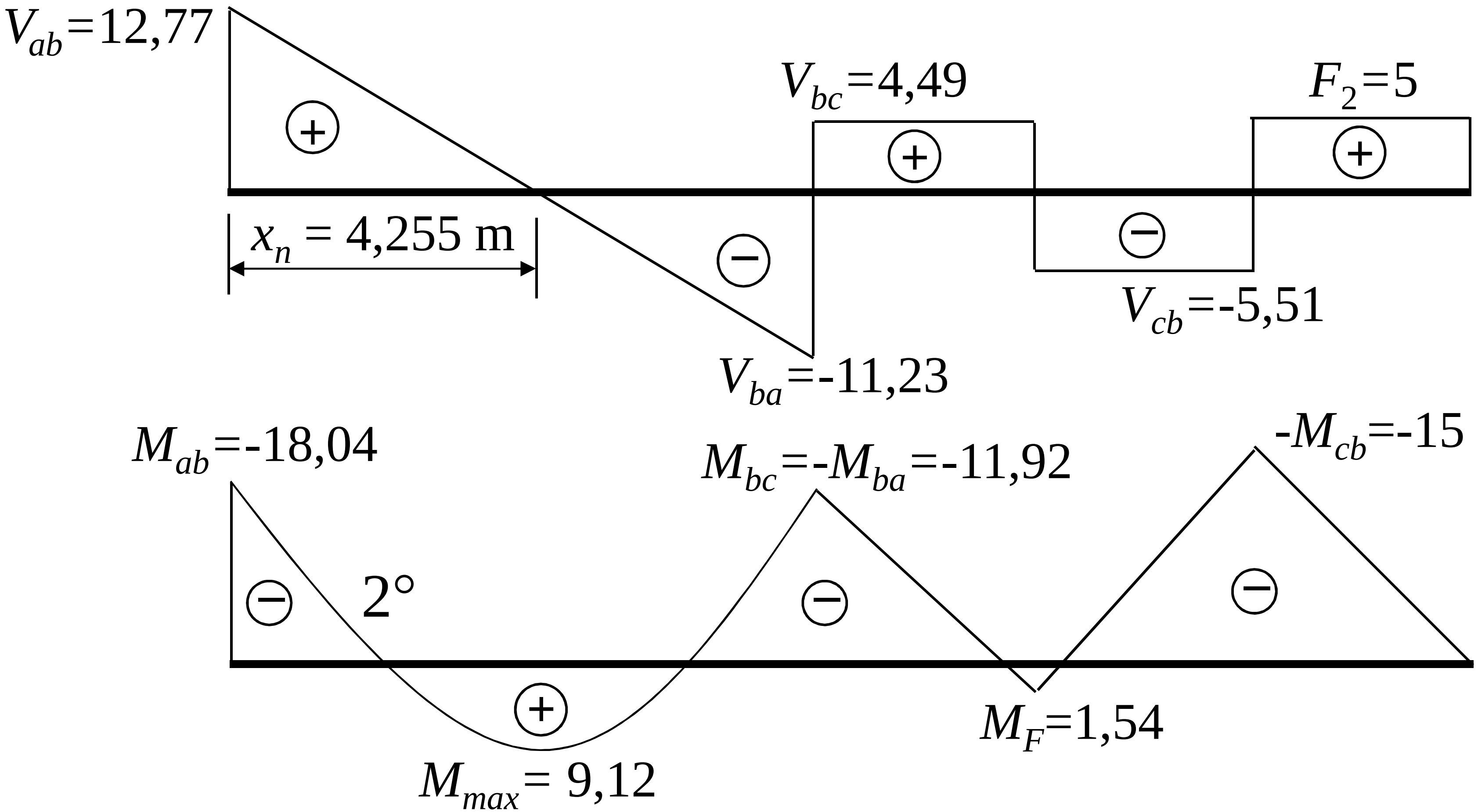
$$R_{az} = R_{ab} = 12{,}77\text{kN}$$

$$R_{bz} = R_{ba} + R_{bc} = 11{,}23 + 4{,}49 = 15{,}72\,\text{kN}$$

$$R_{cz} = R_{cb} + F_2 = 5{,}51 + 5 = 10{,}51\,\text{kN}$$

Slope-Deflection Method: Beams Example 2

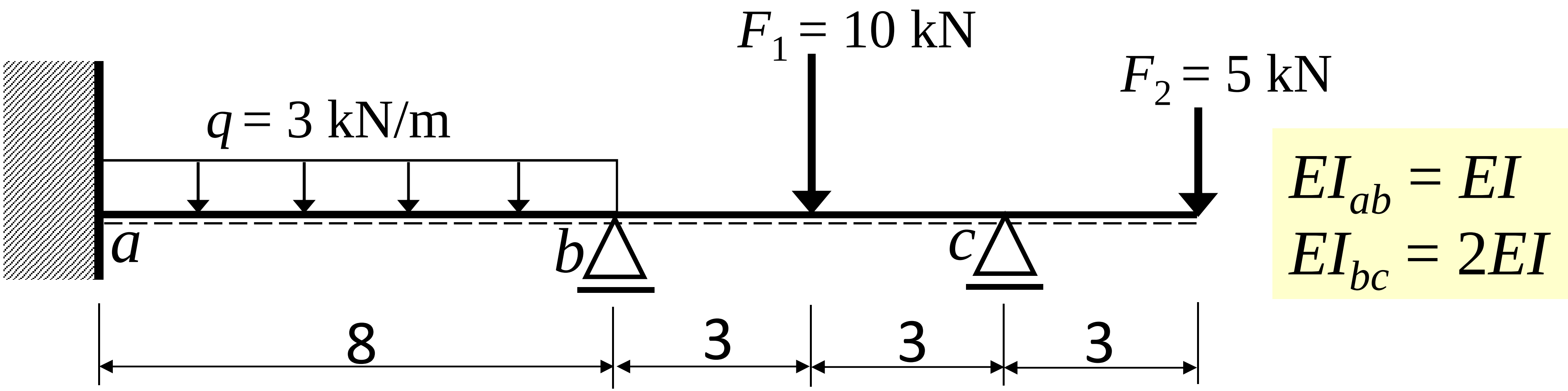
7. Draw shear force and bending moment diagrams.



$$M_{max} = M_{ab} + R_{ab} \cdot x_n - q \cdot \frac{x_n^2}{2} = -18{,}04 + 12{,}77 \cdot 4{,}255 - 3 \cdot \frac{4{,}255^2}{2} = 9{,}12 \text{ kNm}$$

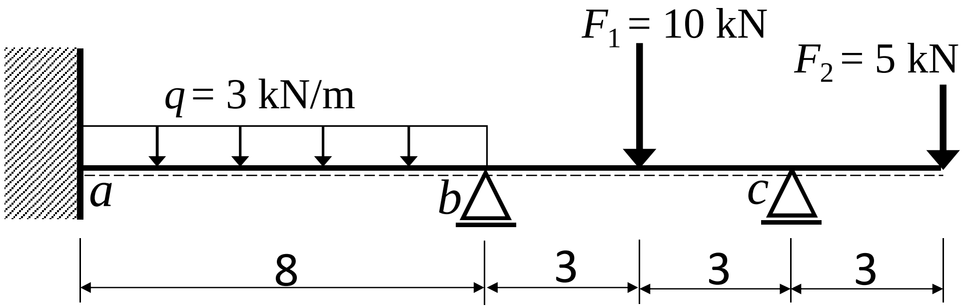
$$M_F = M_{bc} + R_{bc} \cdot \frac{l_{bc}}{2} = -11{,}92 + 4{,}49 \cdot 3 = 1{,}54 \text{ kNm}$$

Slope-Deflection Method: Beams
Example 2, the 2nd possibility of solution



1. Degrees of freedom
   - The continuous beam is kinematically indeterminate to second degree.
     - The 2nd possibility of solution – solve only one unknown joint rotation φ_b ($\varphi_a = 0$, joint rotation φ_c is not necessary to solution because the moment in the cantilever portion M_c is known $\Rightarrow$ beam portion ***bc*** is taken as fixed - hinged).

Slope-Deflection Method: Beams
Example 2, <u>the 2nd possibility of solution</u>

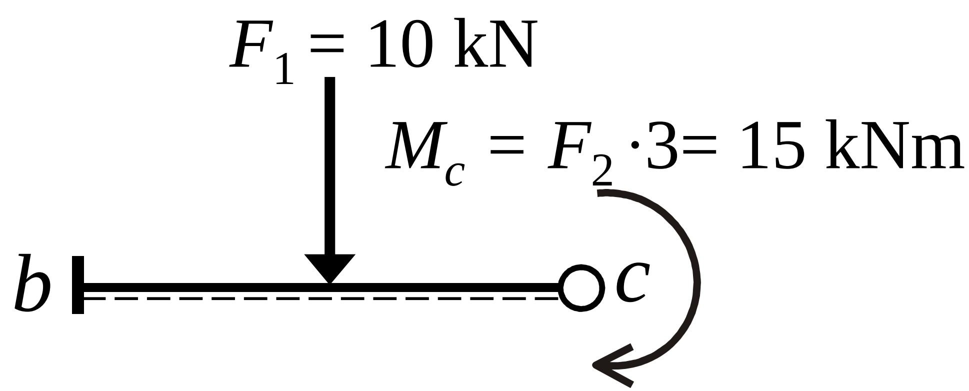


EI = konst., l	$\overline{M}^*_{a,b}$	$\overline{M}^*_{b,a}$	$\overline{M}^*_{b,a}$	$\overline{M}^*_{a,b}$
Q ($l/2$, $l/2$)	$-\frac{1}{8}Ql$	$+\frac{1}{8}Ql$	$+\frac{3}{16}Ql$	$-\frac{3}{16}Ql$
q	$-\frac{1}{12}ql^2$	$+\frac{1}{12}ql^2$	$+\frac{1}{8}ql^2$	$-\frac{1}{8}ql^2$
M (a, b)	$+M\frac{b(2l-3b)}{l^2}$	$+M\frac{a(2l-3a)}{l^2}$	$+M\frac{l^2-3a^2}{2l^2}$	$+M\frac{l^2-3b^2}{2l^2}$

2. Fixed end moments are calculated referring to the table.

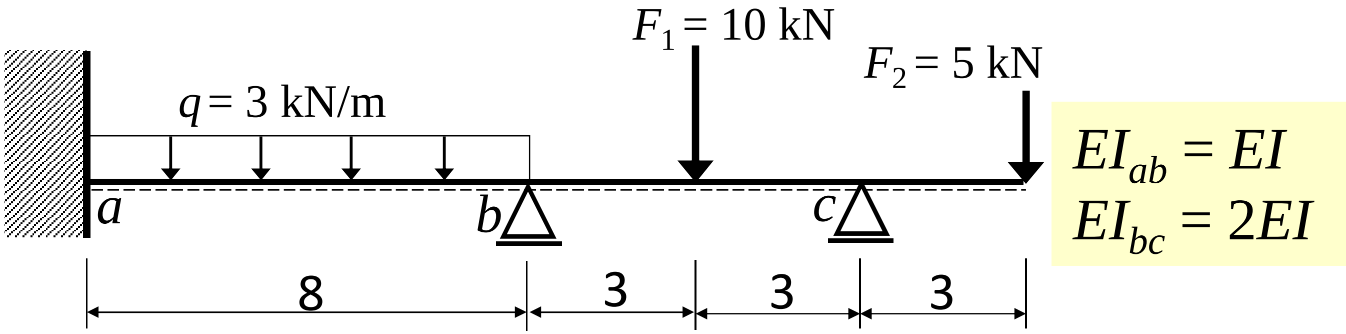
$$\overline{M}_{ab} = -\frac{1}{12}\cdot q\cdot l_{ab}^2 = -\frac{1}{12}\cdot 3\cdot 8^2 = -16\ \text{kNm}$$

$$\overline{M}_{ba} = +\frac{1}{12}\cdot q\cdot l_{ab}^2 = 16\ \text{kNm}$$



$$\overline{M}_{bc} = -\frac{3}{16}\cdot F\cdot l_{bc} + M_c\cdot\frac{l_{bc}^2 - 3\cdot b^2}{2\cdot l_{bc}^2} = -\frac{3}{16}\cdot F\cdot l_{bc} + \frac{M_c}{2} = -\frac{3}{16}\cdot 10\cdot 6 + \frac{15}{2} = -3{,}75\ \text{kNm}$$

Slope-Deflection Method: Beams
Example 2, the 2nd possibility of solution



3. Express internal end moments by slope-deflection equations.

$$M_{ab} = \overline{M}_{ab} + \frac{2EI}{l}\left(2\cdot\varphi_a + \varphi_b\right)$$

$$\boxed{M_{ab} = \overline{M}_{ab} + \frac{3EI}{l}\cdot\varphi_a}$$

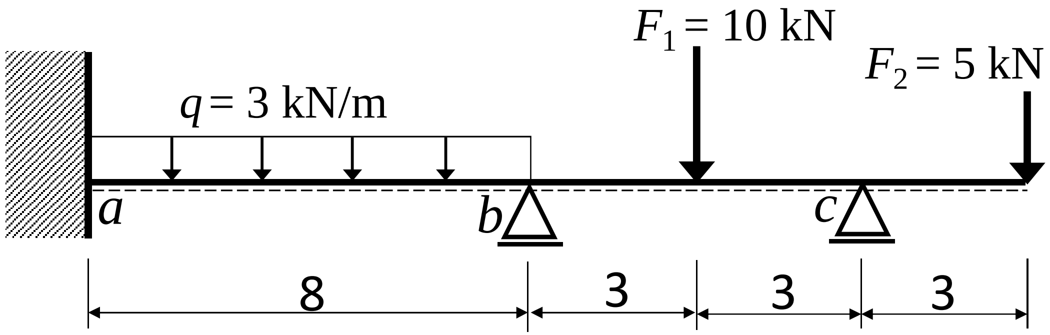
fixed - hinged

$$M_{ab} = \overline{M}_{ab} + \frac{2EI_{ab}}{l_{ab}}\left(2\cdot\varphi_a + \varphi_b\right) = -16 + \frac{2EI}{8}\left(2\cdot 0 + \varphi_b\right) = -16 + \frac{EI}{4}\varphi_b$$

$$M_{ba} = \overline{M}_{ba} + \frac{2EI_{ab}}{l_{ab}}\left(2\cdot\varphi_b + \varphi_a\right) = 16 + \frac{2EI}{8}\left(2\cdot\varphi_b + 0\right) = 16 + \frac{EI}{2}\varphi_b$$

$$M_{bc} = \overline{M}_{bc} + \frac{3EI_{bc}}{l_{bc}}\cdot\varphi_b = -3{,}75 + \frac{3\cdot 2EI}{6}\cdot\varphi_b = -3{,}75 + EI\cdot\varphi_b$$

Slope-Deflection Method: Beams
Example 2, the 2nd possibility of solution



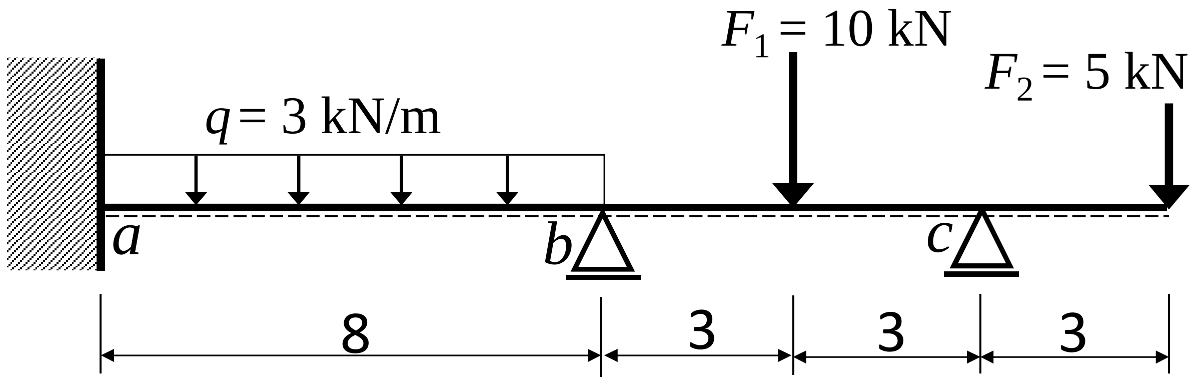
4. Equilibrium equations (write one equilibrium equation for each unknown joint rotation)
   - End moments are expressed in terms of unknown rotation φ_b. Now, the required equation to solve for the rotation φ_b is the moment equilibrium equation at support b.

$$\sum M_b = 0: \quad M_{ba} + M_{bc} = 0$$

$$16 + \frac{EI}{2}\varphi_b - 3{,}75 + EI \cdot \varphi_b = 0$$

$$1{,}5 \cdot EI \cdot \varphi_b + 12{,}25 = 0 \Rightarrow \varphi_b = -\frac{12{,}25}{1{,}5 \cdot EI} = -\frac{8{,}167}{EI}$$

Slope-Deflection Method: Beams
Example 2, the 2nd possibility of solution



5. End moments

- After evaluating φ_b, substitute it to evaluate beam end moments.

$$M_{ab} = -16 + \frac{EI}{4} \cdot \varphi_b = -16 + \frac{EI}{4} \cdot \left(-\frac{8{,}167}{EI} \right) = -18{,}04 \text{ kNm}$$

$$M_{ba} = 16 + \frac{EI}{2} \cdot \varphi_b = 16 + \frac{EI}{2} \cdot \left(-\frac{8{,}167}{EI} \right) = 11{,}92 \text{ kNm}$$

$$M_{bc} = -3{,}75 + EI \cdot \varphi_b = -3{,}75 + EI \cdot \left(-\frac{8{,}167}{EI} \right) = -11{,}92 \text{ kNm}$$

Then the procedure is the same as for the 1st possibility of solution.