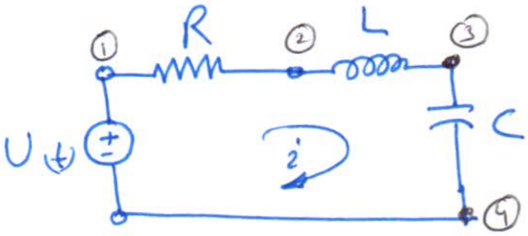
# Examples of the Formulation of Network Equations:

## Example No. 1



K.V. Law requires that

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = V(t)$$

$b - n + 1 = 4 - 4 + 1 = 1$

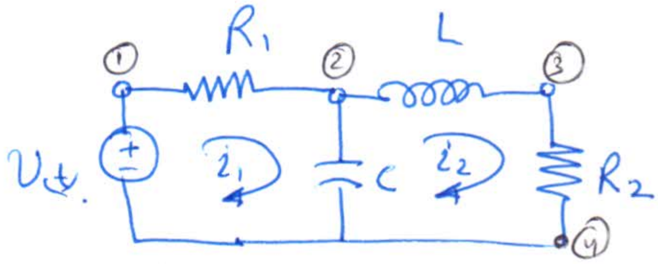
at all times.

This is an integrodifferential equation, which may be changed to a differential equation by differentiation to give;

$$\boxed{L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C} i = \frac{dV(t)}{dt}}$$

Note: Derivatives have been arranged in descending order.

## Example No. 2



No. of equation $= b - n + 1 = 5 - 4 + 1 = 2$.

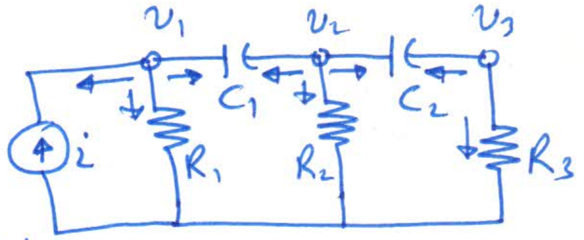
which is also clear from "Window Pane" Rule.

with the two loop currents $i_1$ and $i_2$ assigned with the directions indicated, the equilibrium equations based on Kirchhoff's Voltage Law are;

$$R_1 i_1 + \frac{1}{C}\int_{-\infty}^{t}(i_1 - i_2)\,dt = V(t). \quad \rightarrow (1)$$

$$\frac{1}{C}\int_{-\infty}^{t}(i_2 - i_1)\,dt + L\frac{di_2}{dt} + R_2 i_2 = 0 \quad \rightarrow (2)$$

## Example No.3




A three-node network is shown in Fig: with node-to-datum voltages $v_1$, $v_2$ and $v_3$ assigned as indicated.

Assuming current out of the node to be +ive for each of the nodes in turn gives the 3-K.C. equation

$$\frac{1}{R_1} v_1 + C_1 \frac{d}{dt}(v_1 - v_2) = i(t). \rightarrow ①$$

$$C_1 \frac{d}{dt}(v_2 - v_1) + \frac{1}{R_2} v_2 + C_2 \frac{d}{dt}(v_2 - v_3 = 0 \rightarrow ②$$

$$\frac{1}{R_3} \cdot v_3 + C_2 \frac{d}{dt}(v_3 - v_2) = 0 \rightarrow ③$$

## Example. No.4.

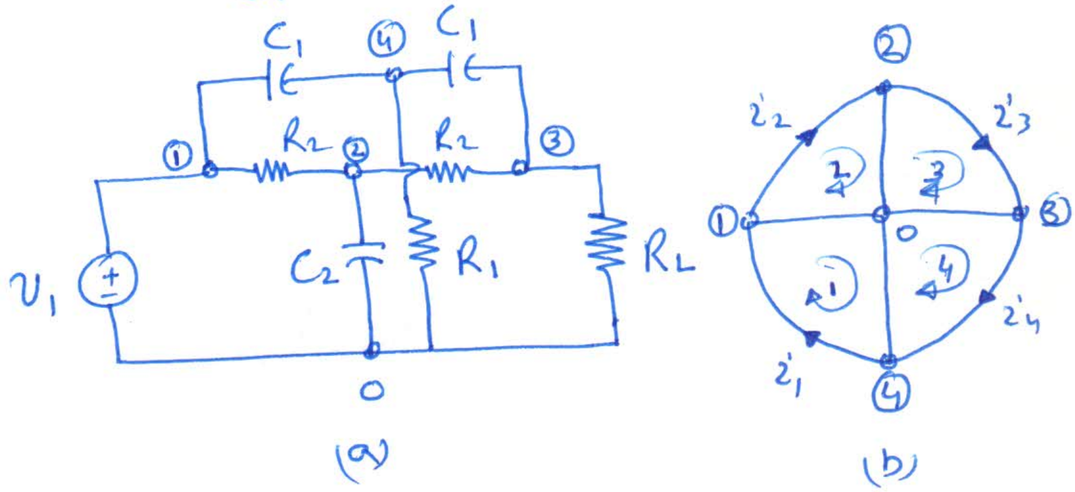



Fig: 3.26. A twin-T, RC network and its graph analyzed in this example.

As the network is more complicated the construction of graph will aid in the formulation of the voltage equation.

$$\frac{1}{C_1}\int i_1 \cdot dt + R_1(i_1 - i_4) = -v_1 \rightarrow ①$$

$$R_2 i_2 + \frac{1}{C_2}\int (i_2 - i_3)\, dt = +v_1 \rightarrow ②$$

$$\frac{1}{C_2}\int (i_3 - i_2)\, dt + R_3 i_3 + R_L(i_3 - i_4) = 0 \rightarrow ③$$

$$\frac{1}{C_1}\int i_4 \cdot dt + R_1(i_4 - i_1) + R_L(i_4 - i_3) = 0 \rightarrow ④$$

Note:
In these examples we have used the integral alone as short hand notation for integral: thus $\int i_k dt = \int_{-\infty}^{t} i_k(t) \cdot dt$.

Example. No. 5

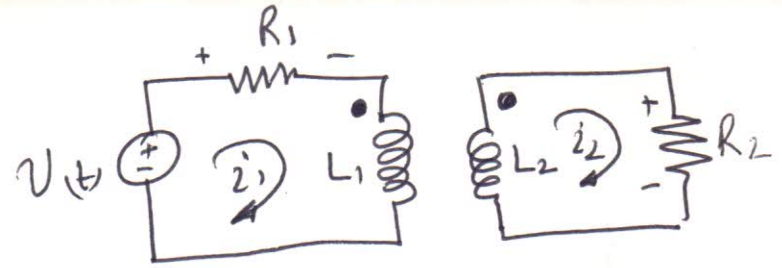



Fig: 3.27: The network of the example containing two parts which are magnetically coupled.

→ This network has two parts which are magnetically coupled. For coupled networks, Eq. (3.12) $(b-n+1)$ must be modified to the form

$$b-n+p$$

where $p$ = number of separate parts of the network.

Similarly the number of node equations for coupled networks is $n-p$ rather than $n-1$.

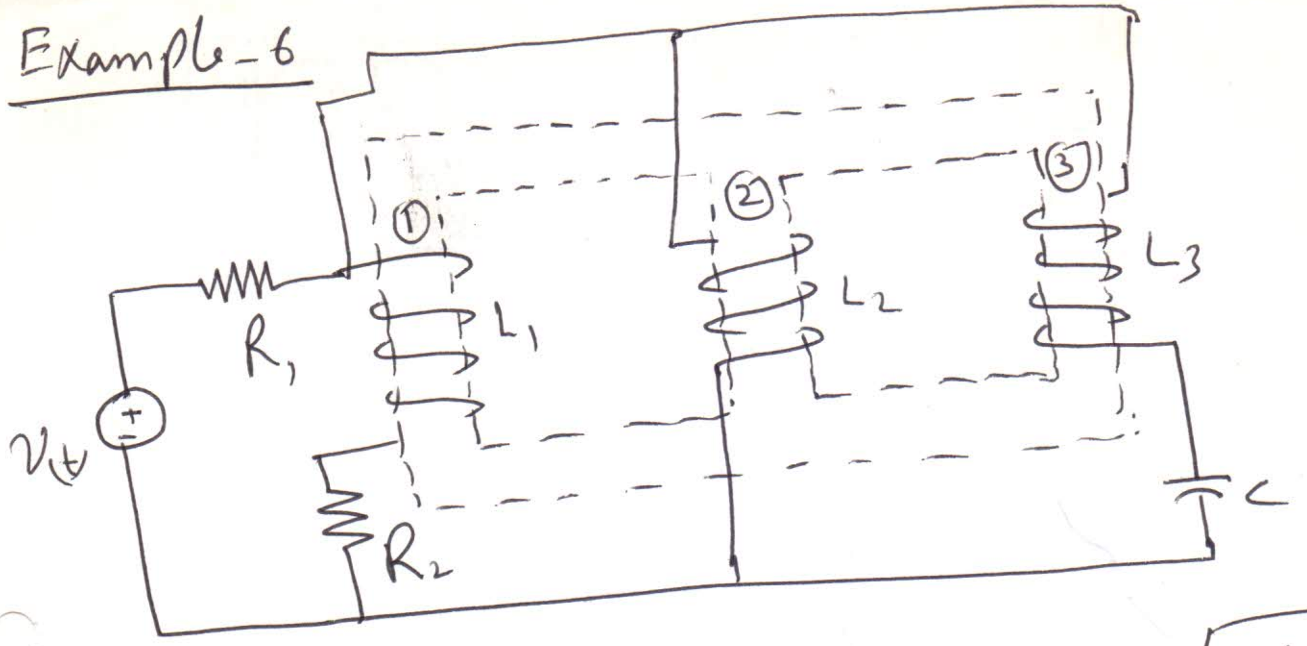
→ Thus for this network no. of loop equations are: $b-n+p = 5-5+2 = 2$.

Equation will be as under:

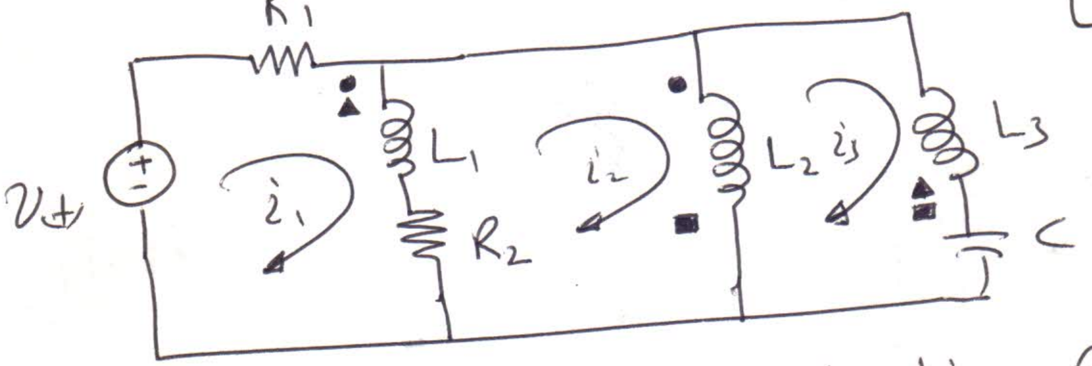
$$R_1 i_1 + L_1 \frac{di_1}{dt} - M\frac{di_2}{dt} = v(t).$$

$$L_2 \frac{di_2}{dt} - M\frac{di_1}{dt} + R_2 i_2 = 0$$

## Example - 6




**K.V.L**

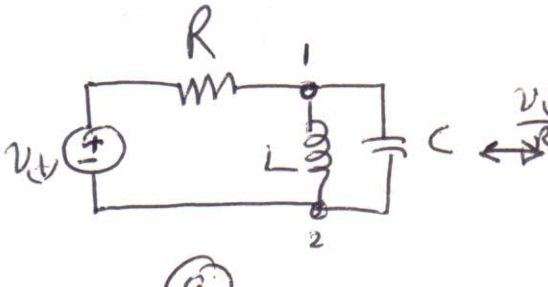



$$R_1 i_1 + L_1 \frac{d(i_1-i_2)}{dt} + M_{12}\frac{d(i_2-i_3)}{dt} - M_{13}\frac{di_3}{dt} + R_2(i_1-i_2) = v(t) \rightarrow (1)$$

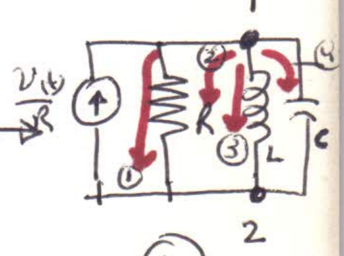
$$R_2(i_2-i_1) + L_1\frac{d(i_2-i_1)}{dt} - M_{12}\frac{d}{dt}(i_2-i_3) + M_{13}\frac{di_3}{dt} + L_2\frac{d(i_2-i_3)}{dt}$$
$$+ M_{21}\frac{d}{dt}(i_1-i_2) + M_{23}\frac{di_3}{dt} = 0 \rightarrow (2)$$

$$L_2\frac{d}{dt}(i_3-i_2) - M_{23}\frac{di_3}{dt} - M_{21}\frac{d}{dt}(i_1-i_2) + L_3\frac{di_3}{dt}$$
$$+ M_{32}\frac{d}{dt}(i_2-i_3) - M_{31}\frac{d}{dt}(i_1-i_2) + \frac{1}{C}\int i_3\,dt = 0 \rightarrow (3)$$

## Example. No. 7




(a)




(b)

Node (2) = Datum Node.

$$i_1 + i_2 + i_3 + i_4 = 0$$

$$-\frac{v(t)}{R} + i_2 + i_3 + i_4 = 0$$

$$\frac{1}{R}v_1 + \frac{1}{L}\int v_1\,dt + C\frac{dv_1}{dt} = \frac{v(t)}{R} \longrightarrow \text{Using fig. (b)}$$

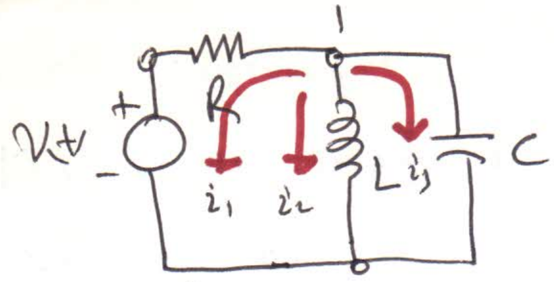



fig: (a)

$$i_1 + i_2 + i_3 = 0$$

$$\frac{1}{R}[V_1 - V(t)] + \frac{1}{L}\int V_1\,dt + C\frac{dV_1}{dt} = 0$$

or

$$\frac{1}{R}V_1 + \frac{1}{L}\int V_1\,dt + C\frac{dV_1}{dt} = \frac{V(t)}{R}$$

This equation is identical to the equation determined from fig: (b)

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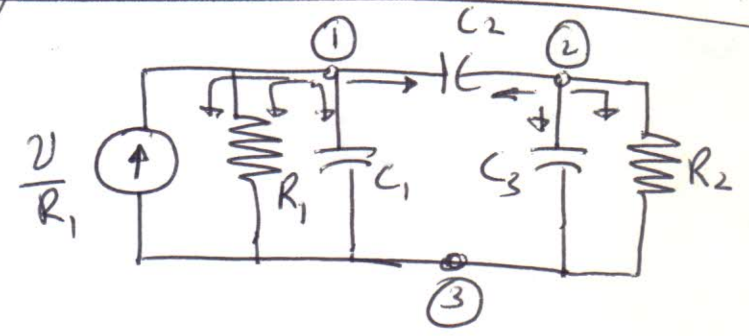
## Example (8)




Node-3 = Datum Node.

Unknown Voltages = node (1) = $v_1$
node (2) = $v_2$

Take: $1/R_1 = G_1$ and $1/R_2 = G_2$.

At node (1)

$$G_1 V = G_1 v_1 + C_1\frac{dv_1}{dt} + C_2\frac{d}{dt}(v_1 - v_2) \longrightarrow (1)$$

At node (2)

$$0 = C_2\frac{d}{dt}(v_2 - v_1) + C_3\frac{dv_2}{dt} + G_2 v_2 \longrightarrow (2)$$

Note: In this Example, formulation on node basis has resulted in fewer differential equations than on the Loop basis.

- It requires less work in solving two simultaneous differential equations than in solving three.
- The Choice of method of formulation, Loop or node, also depends on the objective of analysis.

In this example, if the voltage at node-2 is desired the node method has the advantage over the loop method. But if it is the current flowing in capacitor $C_3$ that is to be found, we must weigh the relative advantages of the two methods.

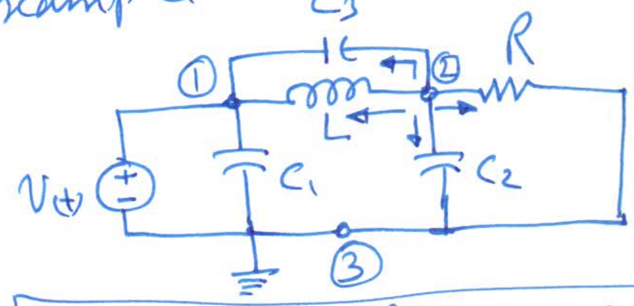
The loop currents can be assigned so that only one loop current flowing in $C_3$, but three simultaneous equations must be solved.

→ Using the node method, we might find the voltage at node-2 first and then determine the current in the capacitor from the equation.

$$i_{C_3} = C_3 \cdot \frac{dV_2}{dt}.$$

The second method involves less computation in this particular example.

Example. (9)




$$\boxed{C_3 \frac{d}{dt}(V_2 - V_1) + \frac{1}{L}\int (V_2 - V_1)\cdot dt + G_2 V_2 + C_2 \frac{dV_2}{dt} = 0}$$

- No series resistance with voltage source.
- 3-independent loops
- 1-Unknown node voltage (2)
- $G_2 = 1/R$.
- $C_1$ does not appear in the equation.
- This is because the voltage at node-1 is independent of the capacitor $C_1$. Capacitor $C_1$ is an extraneous element. $C_1$ may be removed without affecting the network equations.

# Loop Variable Analysis

Thus far we have progressed from the analysis of very simple networks to more complex network configurations for the loop and node method.

→ In the next 3-sections, we will continue the discussion for 3 of the many methods for the formulation of equations to describe networks.

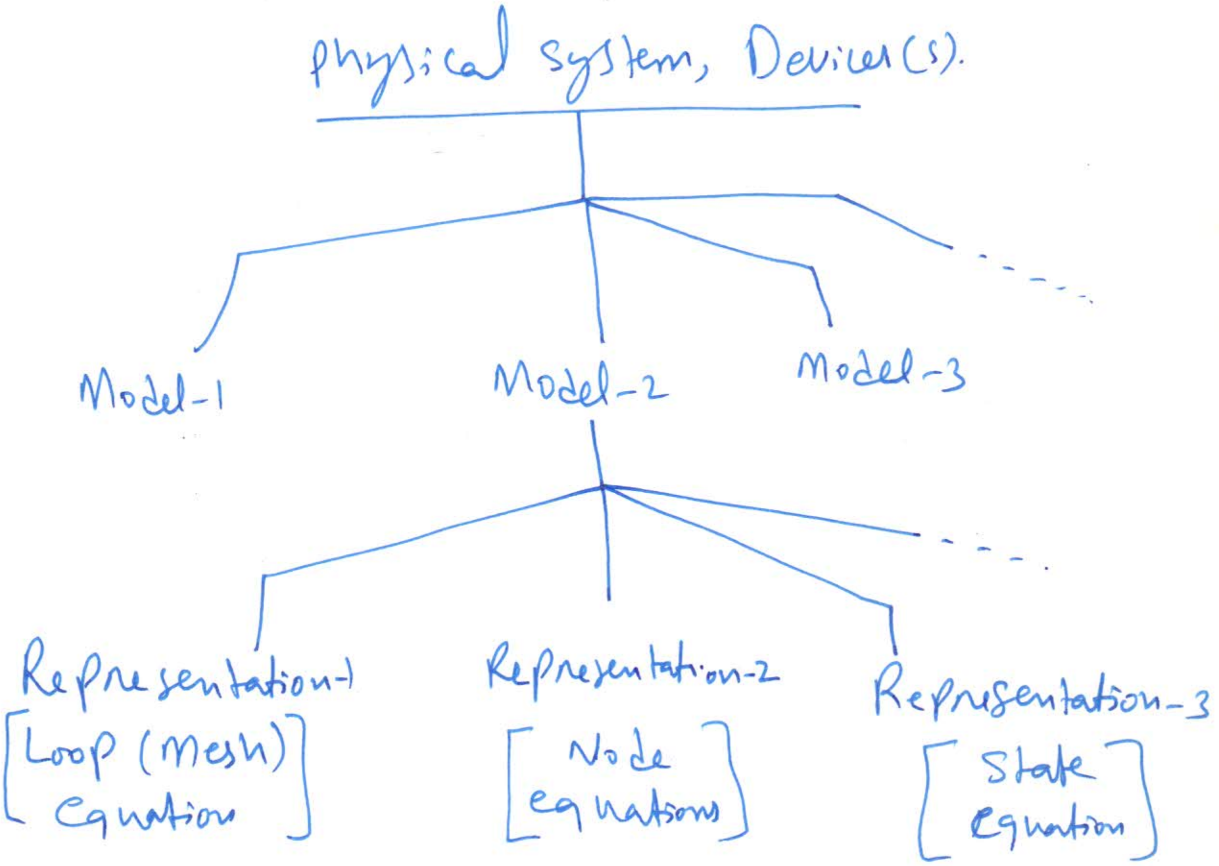



Fig: 3.34. Various representations are available to describe a given model of a physical system.

It is intended to illustrate the point that once we have selected a model of a system of devices, we have a number of alternatives in the representation of that model by a set of network equations.

Factors which come into our choice of representations were already discussed, and they include keeping the number of variables small, finding the desired result as directly as possible and the like.

All valid methods are capable of leading to the same end result, the determination of all branch voltages and branch currents in the network.

→ Now analysis is relatively simple for networks in which there are passive elements only, excluding.

- Mutual inductance
- Controlled sources.

Our approach will be first to consider a simple case and later outline the modifications needed to treat the more general case.

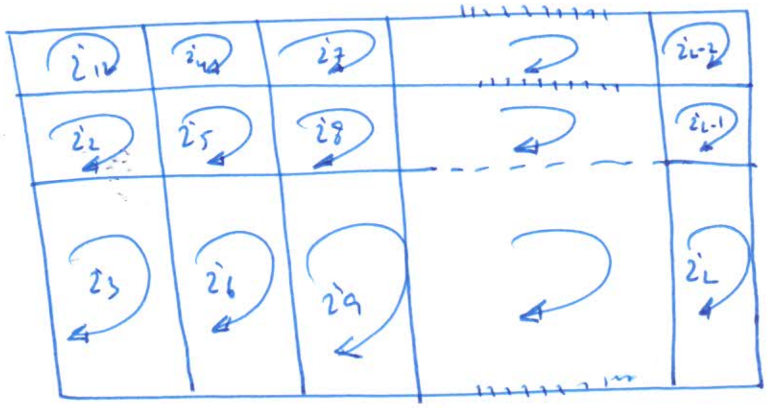



Fig: 3.35. A graph of a network with L- independent loop currents identified.

→ To begin, let us consider an L-loop network, represented by the graph shown in the above figure.

Consider loop-1. This loop may contains, resistance, inductance and Capacitance in any one on all of the branches that make up the loop. Let;

$R_{11}$ be the total resistance in Loop_1

$L_{11}$ be the total inductance in Loop-1

$D_{11}$ be the total elastance in Loop_1

We use elastance instead of Capacitance here because elastance terms add directly for a series circuit, while Capacitance terms combine as

$$\frac{1}{C_{11}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_n}$$

→ There will be Voltage drop in Loop-1 produced by current flow in Loop_2, in Loop-3, Loop-4 - in fact, all Loops in the general Case

→ Rather than specialize on Loop-1, Consider the effect of Currents in the jth Loop on Voltage in Loop_K, where j and K are any integers from 1 to L.

For these two Loops, let

$R_{kj}$ = total resistance Common to Loops K and j

$L_{kj}$ = " inductance (including mutual) " " " " "

$D_{kj}$ = " elastance Common to Loops K and j

The Voltage drop in Loop K produced by Current $i_j$ is

$$R_{kj}\, i_j + L_{kj} \frac{di_j}{dt} + D_{kj} \int i_j \cdot dt$$

OR

$$\left(R_{kj} + L_{kj}\cdot\frac{d}{dt} + D_{kj}\int dt\right) i_j = a_{kj}\cdot i_j.$$

The total voltage drop in loop K will be found by successively considering loop K and the currents flowing in every other loop.

Mathematically this is done by letting j have all values from 1 to L. This total voltage drop must be equal to the total voltage rise from active sources within loop K, which we write as $V_K$.

Then by Kirchhoff's voltage law, we have

$$\sum_{j=1}^{L} a_{kj}\cdot i_j = V_K$$

There remains only to repeat this process for all loops, by letting K have all values from 1-L.

Thus the most general form for K.V. Law for L-loop network is

$$\sum_{j=1}^{L} a_{kj}\, i_j = V_K \qquad \boxed{K = 1, 2, \ldots, L}$$

P.T.O

The expansion of this Concise equation is the following set of equations.

$$a_{11} i_1 + a_{12} i_2 + a_{13} i_3 + \cdots + a_{1L} i_L = v_1$$

$$a_{21} i_1 + a_{22} i_2 + a_{23} i_3 + \cdots + a_{2L} i_L = v_2$$

$$\cdots \qquad \cdots$$

$$a_{L1} i_1 + a_{L2} i_2 + a_{L3} i_3 + \cdots a_{LL} i_L = v_L$$

Arranging these equations in the form of a chart (schedule) are below.

| Eq. | Voltage | $i_1$ | $i_2$ | $i_3$ | $i_4$ | --- | $i_L$ |
|---|---|---|---|---|---|---|---|
| 1 | $v_1$ | $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | ---- | $a_{1L}$ |
| 2 | $v_2$ | $a_{21}$ | $a_{22}$ | $a_{23}$ | $a_{24}$ | ------ | $a_{2L}$ |
| - | --- | -- | -- | -- | -- | -- | -- |
| L | $v_L$ | $a_{L1}$ | $a_{L2}$ | $a_{L3}$ | $a_{L4}$ | ------- | $a_{LL}$ |

If the Loop currents are all assumed +ive is the same Path direction, clockwise for example, then all $a_{jj}$ are +ive and all $a_{jk} (j \neq k)$ are −ive.

In actual problems, of course, many of the operator Coefficients are zero.

The chart we have just written can be written compactly as a matrix equation.

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_L \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1L} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2L} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3L} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & a_{LL} \\ a_{L1} & a_{L2} & a_{L3} & \cdots & \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ \vdots \\ i_L \end{bmatrix}$$

or simply $\quad V = \alpha J.$

[Note: $V$ & $J$ are Column matrices or Vector, $\alpha$ is a square matrix]

$$v_1 = a_{11} i_1 + a_{12} i_2 + a_{13} i_3 + \cdots + a_{1L} i_L.$$

which is Eq (3-46) $\rightarrow$ $\sum_{j=1}^{L} a_{kj} \cdot i_j = v_k, \quad k = 1, 2, \ldots, L$

for $k = 1$.

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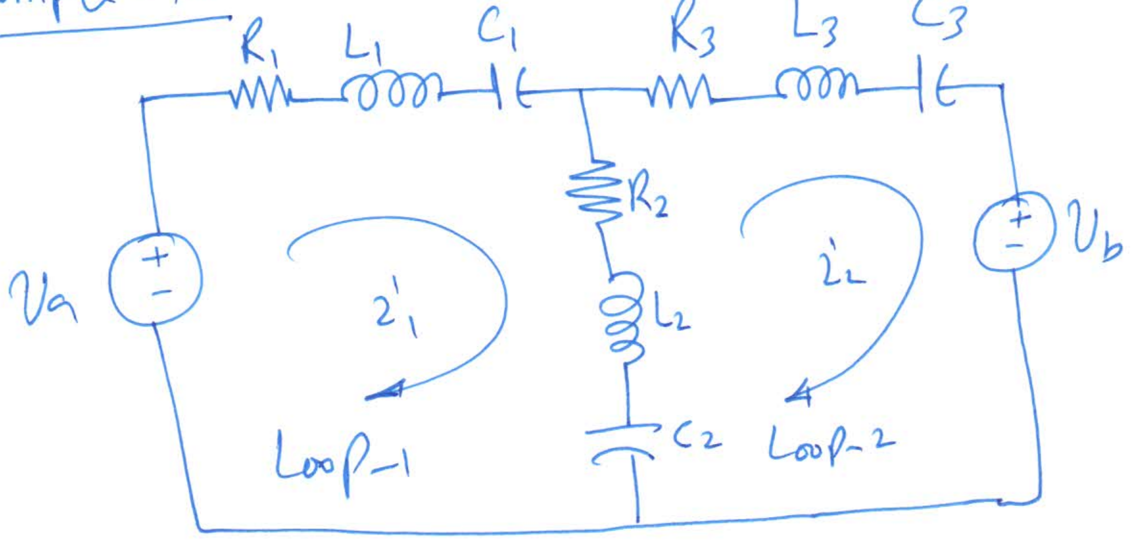
Example - 10




Kirchhoff's Voltage Law.

$$\sum_{j=1}^{2} a_{kj} i_j = v_k \qquad k = 1, 2$$

or in expanded form

$$a_{11} i_1 + a_{12} i_2 = v_1$$
$$a_{21} i_1 + a_{22} i_2 = v_2$$

The operator coefficients are found by inspection of the network as follows.

$$a_{11} = (R_1 + R_2) + (L_1 + L_2)\frac{d}{dt} + (D_1 + D_2)\int dt.$$

$$a_{22} = (R_2 + R_3) + (L_2 + L_3)\frac{d}{dt} + (D_2 + D_3)\int dt.$$

$$a_{12} = a_{21} = -\left(R_2 + L_2 \frac{d}{dt} + D_2 \int dt\right)$$

$$\boxed{v_1 = V_a} \quad \text{and} \quad \boxed{v_2 = -V_b}$$

# Node-Variable Analysis

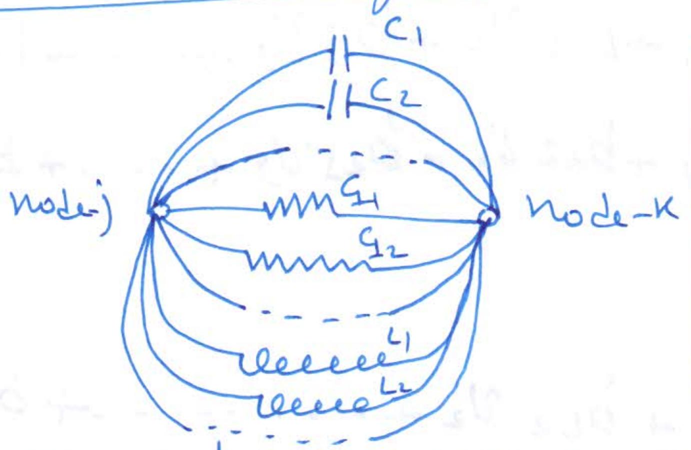



Fig: 3.37.

Elements connecting nodes j & K. The 3-kinds of elements may be combined to give an equivalent Parallel RLC network between nodes j and K

Consider a network with n-nodes and only one part. There will be n-1 independent node pairs.

we will select the node-to-datum Voltages as our variables exclusively.

The form of the voltages for branch connecting node j to node K with j positive will be

$V_j - V_K$ (from K.V.Law).

For each of the n-1 nodes at which the K.C. Law will be formulated, we will assume that currents are directed out of the node to be consistent with the Voltage sign assignment we have just made

$\sum i = 0$ K.C.L.

$i_1 + i_2 + i_3 + i_4 = 0$


we will follow the practice of converting all voltage sources into equivalent current sources as preparation

$$\begin{bmatrix} b_{11}v_1 + b_{12}v_2 + b_{13}v_3 + \cdots + b_{1L}v_L \\ b_{21}v_1 + b_{22}v_2 + b_{23}v_3 + \cdots + b_{2L}v_L \\ \vdots \\ b_{L1}v_1 + b_{L2}v_2 + \cdots + b_{LL}v_L \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ \vdots \\ i_L \end{bmatrix}$$

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1L} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2L} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{L1} & b_{L2} & b_{L3} & \cdots & b_{LL} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_2 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_L \\ \vdots \\ i_L \end{bmatrix}$$

of the network preceding the writing of the equations.

Let us postpone consideration of mutual inductance and controlled sources, and consider a passive network made up of resistors, capacitors and inductors.

Note first that for elements connected as shown in fig. 3.37, the elements may be replaced by an equivalent system made up as follows:

1. All parallel capacitances replaced by an equivalent capacitance of value

$$C_{kj} = C_1 + C_2 + \cdots$$

2. An equivalent resistance ~~capacitance~~ found by adding conductances as

$$G_{kj} = \frac{1}{R_{kj}} = G_1 + G_2 + \cdots$$

3. An equivalent inductance of value $L_{kj}$, when

$$\frac{1}{L_{kj}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots$$

Applying this network simplification to the elements from node k to all other nodes from j = 1 to j = N, we have the equation

$$\sum_{j=1}^{N} \left( G_{kj} + C_{kj} \cdot \frac{d}{dt} + \frac{1}{L_{kj}} \int dt \right) V_j = i_k \qquad (k = 1, 2, \ldots)$$

which may be written concisely as

$$\sum_{j=1}^{N} b_{kj} \cdot V_j = i_k \qquad k = 1, 2, \ldots N.$$

by letting: $b_{kj}$ · summarize the operations.

$$b_{kj} = \left( G_{kj} + C_{kj} \frac{d}{dt} + \frac{1}{L_{kj}} \int dt \right)$$

The expansion of above equation has the same form as expansion for the loop case with "a" is replaced by "b"'s, "i"'s by "v"'s and "v"'s by "i"'s.

In applying this equation to networks, it is not necessary to simplify the network by combining elements. At node j, the capacitance $\underline{C_{jj}}$ is the sum of the capacitance connected to node j or the capacitance from node j to ground with all other nodes grounded.

The value of $C_{kj}$ is the sum of capacitances connected between node j and node k or the capacitance from node j to node k with all other nodes grounded.
Similarly instructions hold for inverse inductance $1/L$ and for conductance $G = 1/R$.

Coefficients can thus be found by inspection by simply noting which elements are "hanging on" or "hanging between" the various nodes.

If the same convention for the current is maintained in formulating all node equations for a network, the sign of $b_{kj}$ will be positive when $k=j$ and negative when $k \neq j$

## Example - 12

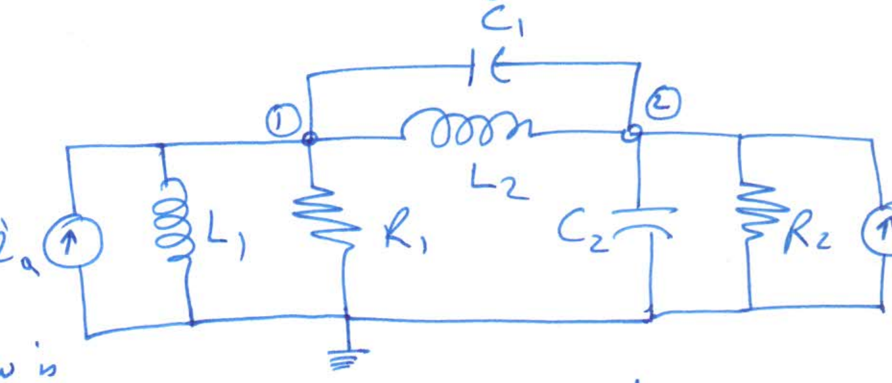



Fig: Network with two independent node-pair voltages analysed.

For this network, K.C. Law is

$$\sum_{j=1}^{2} b_{kj} v_j = i_k \quad k = 1,2$$

or

$$b_{11} V_1 + b_{12} V_2 = i_1$$
$$b_{21} V_1 + b_{22} V_2 = i_2$$

In matrix form,

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

Gen:

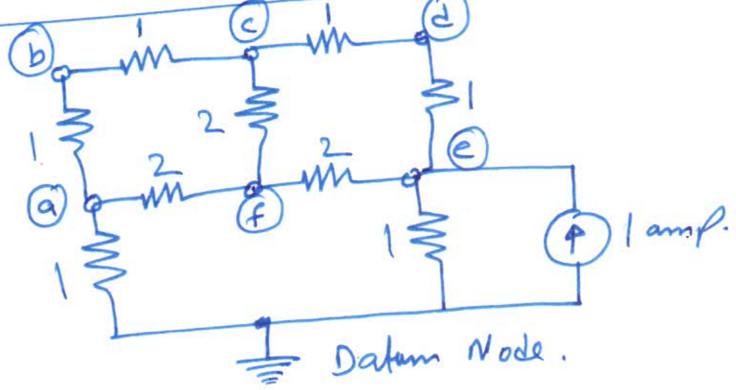
$$b_{11} V_1 + b_{12} V_2 + b_{13} V_3 + \cdots + b_{1N} V_N = I_1$$
$$b_{21} V_1 + b_{22} V_2 + b_{23} V_3 + \cdots + b_{2N} V_N = I_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$b_{N1} V_1 + b_{N2} V_2 + b_{N3} V_3 + \cdots + b_{NN} V_N = I_N$$

values for the operator coefficients are summarized in chart form as follows.

| Eq. | Current | $V_1$ | $V_2$ |
|---|---|---|---|
| 1 | $i_a$ | $G_1 + C_1 \frac{d}{dt} + \left(\frac{1}{L_1} + \frac{1}{L_2}\right) \int dt$ | $- C_1 \frac{d}{dt} - \frac{1}{L_2} \int dt$ |
| 2 | $i_b$ | $- C_1 \frac{d}{dt} - \frac{1}{L_2} \int dt$ | $+ G_2 + (C_1 + C_2) \frac{d}{dt} + \frac{1}{L_2} \int dt$ |

$$\begin{bmatrix} i_a \\ \\ i_b \end{bmatrix} = \begin{bmatrix} G_1 + C_1\frac{d}{dt} + \left(\frac{1}{L_1}+\frac{1}{L_2}\right)\int dt & -C_1\frac{d}{dt} - \frac{1}{L_2}\int dt \\ \\ -C_1\frac{d}{dt} - \frac{1}{L_2}\int dt & + G_2 + (C_1+C_2)\frac{d}{dt} + \frac{1}{L_2}\int dt \end{bmatrix}\begin{bmatrix} v_1 \\ \\ v_2\end{bmatrix}$$

## Example: 13




① 7-nodes including datum Node

② elements values are in ohms

③ six-node variable equations may be routinely written in the following chart form.

| Eq. for node | current | $v_a$ | $v_b$ | $v_c$ | $v_d$ | $v_e$ | $v_f$ |
|---|---|---|---|---|---|---|---|
| | | coefficient of | | | | | |
| a | 0 | $\frac{5}{2}$ | $-1$ | 0 | 0 | 0 | $-1/2$ |
| b | 0 | $-1$ | 2 | $-1$ | 0 | 0 | 0 |
| c | 0 | 0 | $-1$ | $5/2$ | $-1$ | 0 | $-1/2$ |
| d | 0 | 0 | 0 | $-1$ | 2 | $-1$ | 0 |
| e | 1 | 0 | 0 | 0 | $-1$ | $5/2$ | $-1/2$ |
| f | 0 | $-1/2$ | 0 | $-1/2$ | 0 | $-1/2$ | $3/2$ |

or in matrix form

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5/2 & -1 & 0 & 0 & 0 & -1/2 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 5/2 & -1 & 0 & -1/2 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 5/2 & -1/2 \\ -1/2 & 0 & -1/2 & 0 & -1/2 & 3/2 \end{bmatrix}\begin{bmatrix} v_a \\ v_b \\ v_c \\ v_d \\ v_e \\ v_f \end{bmatrix}$$

$$[i] = [Y][v]$$

Such equations can be written by inspection using the "hanging on" and "hanging between" rule and the sign convention for the ij and Gkj entries.

Note: that all terms of the principle diagonal are +ive and that symmetry exists with respect to the principal diagonal.

- Special problems are encountered in the nodal analysis of networks containing mutual inductance, and a good working rule is to bypass the problem by always analyzing such networks on loop basis.

- Should nodal analysis be required, one approach is to replace the coupled coils by an equivalent network without mutual inductance.

- The presence of controlled sources in the network to be analyzed creates no special problems but generally results in a non-symmetrical matrix of the form given in above equation.

## Determinants: Minors and the Gauss Elimination method.

The array of quantities enclosed by straight line brackets is known as Determinant of order n

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \longrightarrow \text{elements:}$$

Quantities in horizontal lines form row, and quantities in vertical lines form columns.
Such a determinant is square, having $n$ rows and $n$- columns.
Each of the $n^2$ quantities in the determinant is known as an element. Each element position in determinant is identified by a double subscript, the first subscript indicating row and second indicating column.

$$a_{\underset{\text{row}}{2}\,\overset{}{1}\rightarrow\text{column.}}$$

Elements along the line extending from $a_{11}$ to $a_{nn}$ form the principal diagonal of the determinant.
A determinant has a value which is a function of the values of its elements. In finding this value, we must make use of rules for expansion of the determinants in terms of the elements.

2nd and 3rd order determinants have expansion that are familiar from studies in elementary algebra.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

and

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}\cancel{(}a_{21}a_{32} - a_{13}a_{22}a_{31}.$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{23}\cdot a_{32}\cdot a_{11} - a_{33}a_{21}a_{12}.$$

Expansions for determinants of order higher than 3rd are conveniently made in terms of minors.

Minor of any element of a determinant $a_{jk}$ is the determinant which remains when the column and row containing $a_{jk}$ are deleted.

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

the minor of $a_{11}$, for example, is

**Minor**

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

A minor of the element $a_{jk}$ multiplied by $(-1)^{j+k}$ is given the name Cofactor.

**ofactor**

$$\text{Cofactor} = (-1)^{j+k}\,\underbrace{(\text{minor})}_{a_{jk}}$$

$$\Delta_{jk} = (-1)^{j+k} M_{jk}.$$

Expansion of a determinant in terms of minors (or cofactors) consists of successive reduction of determinant order.

determinant of order $n$ is equal to the sum f the Product of the elements of any row or olumn multiplied by their corresponding $(n-1)$ rder co-factors.

plying this rule to the expansion of the determinant above determinant along the Ist. Column gives

$$A = a_{11} M_{11} - a_{21} M_{21} + a_{31} \cdot M_{31}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

There are $2n$ equivalent expansions of the determinant about the n-rows or n-columns. The minor determinants can, in turn, be expanded by the same rule and the process continued until the value of $\Delta$ is given as the sum of $n \times n!$ product factors.

The facts about determinants that we have just reviewed are essential in solving simultaneous equations of the form:

$$a_{11} i_1 + a_{12} i_2 + a_{13} i_3 + \cdots + a_{1L} \cdot i_L = V_1$$

$$- - - - - - - - - - - - - - -$$

$$a_{L1} i_1 + a_{L2} i_2 + a_{L3} \cdot i_3 + \cdots + a_{LL} \cdot i_L = V_L.$$

that have resulted from application of K.V. law (and similar equations from K.C. law).

The solution to such simultaneous equations is given by Cramer's rule as

$$i_1 = \frac{D_1}{\Delta}, \quad i_2 = \frac{D_2}{\Delta}, \cdots \; i_L = \frac{D_L}{\Delta}$$

where $\Delta$ is the system determinant given as

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1L} \\ a_{21} & a_{22} & \cdots & a_{2L} \\ - & - & - & - \\ a_{L1} & a_{L2} & \cdots & a_{LL} \end{vmatrix}$$

which must be different from zero for the solution $i_1, i_2, \cdots, i_n$ to be unique, and $D_j$ is the determinant formed by replacing the jth Column of a coefficient by the column $v_1, v_2 \cdots v_n$.

$$i_1 = \frac{\begin{vmatrix} v_1 & a_{12} & \cdots & a_{1L} \\ v_2 & a_{22} & \cdots & a_{2L} \\ \cdots & \cdots & \cdots & \cdots \\ v_L & a_{L2} & \cdots & a_{LL} \end{vmatrix}}{\Delta}$$

for the above 3X3 Matrix Case

co-factors.

$$i_1' = \frac{\begin{vmatrix} v_1 & a_{12} & a_{13} \\ v_2 & a_{22} & a_{23} \\ v_3 & a_{32} & a_{33} \end{vmatrix}}{\Delta} = \frac{v_1 \Delta_{11} + v_2 \Delta_{21} + v_3 \Delta_{31}}{\Delta}$$

or

$$i_1' = \frac{\Delta_{11}}{\Delta} \cdot v_1 + \frac{\Delta_{21}}{\Delta} v_2 + \frac{\Delta_{31}}{\Delta} \cdot v_3.$$

Similarly

$$i_2' = \frac{\begin{vmatrix} a_{11} & v_1 & a_{13} \\ a_{21} & v_2 & a_{23} \\ a_{31} & v_3 & a_{33} \end{vmatrix}}{\Delta} = \frac{v_1 \Delta_{12} + v_2 \Delta_{22} + v_3 \Delta_{32}}{\Delta}$$

or

$$i_2' = \frac{\Delta_{12}}{\Delta} \cdot v_1 + \frac{\Delta_{22}}{\Delta} \cdot v_2 + \frac{\Delta_{32}}{\Delta} \cdot v_3.$$

and

$$i_3' = \frac{\Delta_{13}}{\Delta} \cdot v_1 + \frac{\Delta_{23}}{\Delta} \cdot v_2 + \frac{\Delta_{33}}{\Delta} \cdot v_3.$$

The form of these equations is greatly simplified if all V's except one are zero, corresponding to only one driving voltage source

Example: 14 For a certain 3-loop network, the following equations are given;

$$5i_1 - 2i_2 - 3i_3 = 10$$
$$-2i_1 + 4i_2 - 1i_3 = 0$$
$$-3i_1 - 1i_2 + 6i_3 = 0$$

From Cramer's Rule we write the solution for $i_1$ as.

$$i_1 = \frac{D_1}{\Delta} = \frac{10\begin{vmatrix} 4 & -1 \\ -1 & 6 \end{vmatrix} - 0\begin{vmatrix} -2 & -3 \\ -1 & 6 \end{vmatrix} + 0\begin{vmatrix} -2 & -3 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 5 & -2 & -3 \\ -2 & 4 & -1 \\ -3 & -1 & 6 \end{vmatrix}}$$

$$= \frac{230}{43}$$

Similarly:

$$i_2 = \frac{-(+10)\begin{vmatrix} -2 & -1 \\ -3 & 6 \end{vmatrix}}{\Delta} = \frac{150}{43}$$

$$i_3 = \frac{+(10)\begin{vmatrix} -2 & 4 \\ -3 & -1 \end{vmatrix}}{\Delta} = \frac{140}{43}.$$

P.T.O